Universitext

R. Balakrishnan
K. Ranganathan

A Textbook of Graph Theory

Second Edition

Springer

Universitext

Universitext

Universitext is a series of textbooks that presents material from a wide variety of mathematical disciplines at master's level and beyond. The books, often well class-tested by their author, may have an informal, personal even experimental approach to their subject matter. Some of the most successful and established books in the series have evolved through several editions, always following the evolution of teaching curricula, to very polished texts.

Thus as research topics trickle down into graduate-level teaching, first textbooks written for new, cutting-edge courses may make their way into *Universitext*.

For further volumes:
http://www.springer.com/series/223

R. Balakrishnan • K. Ranganathan

A Textbook of Graph Theory

Second Edition

Springer

R. Balakrishnan
Department of Mathematics
Bharathidasan University
Tiruchirappalli, India

K. Ranganathan
Deceased

ISSN 0172-5939 ISSN 2191-6675 (electronic)
ISBN 978-1-4614-4528-9 ISBN 978-1-4614-4529-6 (eBook)
DOI 10.1007/978-1-4614-4529-6
Springer New York Heidelberg Dordrecht London

Library of Congress Control Number: 2012946176

Mathematics Subject Classification: 05Cxx

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface to the Second Edition

As I set out to prepare this Second Edition, I realized that I missed very much my coauthor K. Ranganathan, who had an untimely death in 2002; but then his guiding spirit was always there to get me going.

This Second Edition is a revised and enlarged edition with two new chapters—one on domination in graphs (Chap. 10) and another on spectral properties of graphs (Chap. 11)—and an enlarged chapter on graph coloring (Chap. 7). Chapter 10 presents the basic properties of the domination number of a graph and also deals with Vizing's conjecture on the domination number of the Cartesian product of two graphs. Chapter 11 contains several results on the eigenvalues of graphs and includes a section on the Ramanujan graphs and another on the energy of graphs. The new additions in Chap. 7 include the introduction of b-coloring in graphs and an extension of the discussion of the Myceilskian of a graph over what was given in the First Edition. The sections of Chap. 10 of the First Edition that contained some applications of graph theory have been shifted in the Second Edition to the relevant chapters: "The Connector Problems" to Chap. 4, "The Timetable Problem" to Chap. 5 and the "Application to Social Psychology" to Chap. 1.

There are many who helped me to bring out this Second Edition. First and foremost, I owe my thanks to my former colleague S. Baskaran, who class-tested most of the First Edition, pointed out errors, and came up with many useful suggestions. My thanks are also due to Ashwin Ganesan, S. Francis Raj, P. Paulraja, and N. Sridharan, who read portions of the book; R. Sampathkumar, who proofread most of this edition; and A. Anuradha, who fixed all the figures and consolidated the entire material. Typesetting in LaTeX was done by Mohammed Parvees and R. Sampathkumar, and it is my pleasure to thank them.

I welcome any comments, suggestions, and corrections from readers. They can be sent to me at the email address: mathbala@sify.com.

It was a pleasure working with Springer New York, especially Kaitlin Leach, who was in charge of publishing this edition.

Tiruchirappalli, Tamil Nadu, India R. Balakrishnan

Preface to the First Edition

Graph theory has witnessed an unprecedented growth in the 20th century. The best barometer to indicate this growth is the explosion in the number of pages that section 05: Combinatorics (in which the major share is taken by graph theory) occupies in the *Mathematical Reviews.* One of the main reasons for this growth is the applicability of graph theory in many other disciplines, such as physics, chemistry, psychology, and sociology. Yet another reason is that some of the problems in theoretical computer science that deal with complexity can be transformed into graph-theoretical problems.

This book aims to provide a good background in the basic topics of graph theory. It does not presuppose deep knowledge of any branch of mathematics. As a basic text in graph theory, it contains, for the first time, Dirac's theorem on k-connected graphs (with adequate hints), Harary–Nash–Williams' theorem on the hamiltonicity of line graphs, Toida–McKee's characterization of Eulerian graphs, the Tutte matrix of a graph, David Sumner's result on claw-free graphs, Fournier's proof of Kuratowski's theorem on planar graphs, the proof of the nonhamiltonicity of the Tutte graph on 46 vertices, and a concrete application of triangulated graphs.

An ambitious teacher can cover the entire book in a one-year (equivalent to two semesters) master's course in mathematics or computer science. However, a teacher who wants to proceed at leisurely pace can omit the sections that are starred. Exercises that are starred are nonroutine.

The book can also be adapted for an undergraduate course in graph theory by selecting the following sections: 1.1–1.6, 2.1–2.3, 3.1–3.4, 4.1–4.5, 5.1–5.4, 5.5 (omitting consequences of Hall's theorem), 5.5 (omitting the Tutte matrix), 6.1–6.3, 7.1, 7.2, 7.5 (omitting Vizing's theorem), 7.8, 8.1–8.4, and Chap. 10.

Several people have helped us by reviewing the manuscript in parts and offering constructive suggestions: S. Arumugam, S. A. Choudum, P. K. Jha, P. Paulraja, G. Ramachandran, S. Ramachandran, G. Ravindra, E. Sampathkumar, and R. Sampathkumar. We thank all of them most profusely for their kindness in sparing for our sake a portion of their precious time. Our special thanks are due to P. Paulraja and R. Sampathkumar, who have been a constant source of inspiration to us ever since we started working on this book rather seriously. We also thank D. Kannan, Department of Mathematics, University of Georgia, for reading the manuscript and suggesting some stylistic changes.

We also take this opportunity to thank the authorities of our institutions, Annamalai University, Annamalai Nagar, and National College, Tiruchirappalli, for their kind encouragement. Finally, we thank the University Grants Commission, Government of India, for its financial support for writing this book.

Our numbering scheme for theorems and exercises is as follows. Each exercise bears two numbers, whereas each theorem, lemma, and so forth bears three numbers. Therefore, Exercise 3.4 is the fourth exercise of Sect. 3 of a particular chapter, and Theorem 6.6.1 is the first result of Sect. 6 of Chap. 6.

Tiruchirappalli, Tamil Nadu, India

R. Balakrishnan
K. Ranganathan

Contents

1	**Basic Results**	1
1.1	Introduction	1
1.2	Basic Concepts	1
1.3	Subgraphs	8
1.4	Degrees of Vertices	10
1.5	Paths and Connectedness	13
1.6	Automorphism of a Simple Graph	18
1.7	Line Graphs	20
1.8	Operations on Graphs	24
1.9	Graph Products	26
1.10	An Application to Chemistry	31
1.11	Application to Social Psychology	31
1.12	Miscellaneous Exercises	34
	Notes	35
2	**Directed Graphs**	37
2.1	Introduction	37
2.2	Basic Concepts	37
2.3	Tournaments	39
2.4	k-Partite Tournaments	42
2.5	Exercises	47
	Notes	47
3	**Connectivity**	49
3.1	Introduction	49
3.2	Vertex Cuts and Edges Cuts	49
3.3	Connectivity and Edge Connectivity	53
3.4	Blocks	59
3.5	Cyclical Edge Connectivity of a Graph	61
3.6	Menger's Theorem	61
3.7	Exercises	70
	Notes	71

4 **Trees** ... 73
4.1 Introduction ... 73
4.2 Definition, Characterization, and Simple Properties ... 73
4.3 Centers and Centroids ... 77
4.4 Counting the Number of Spanning Trees ... 81
4.5 Cayley's Formula ... 84
4.6 Helly Property ... 86
4.7 Applications ... 87
4.7.1 The Connector Problem ... 87
4.7.2 Kruskal's Algorithm ... 88
4.7.3 Prim's Algorithm ... 90
4.7.4 Shortest-Path Problems ... 92
4.7.5 Dijkstra's Algorithm ... 92
4.8 Exercises ... 94
Notes ... 95

5 **Independent Sets and Matchings** ... 97
5.1 Introduction ... 97
5.2 Vertex-Independent Sets and Vertex Coverings ... 97
5.3 Edge-Independent Sets ... 99
5.4 Matchings and Factors ... 100
5.5 Matchings in Bipartite Graphs ... 104
5.6 Perfect Matchings and the Tutte Matrix ... 112
Notes ... 115

6 **Eulerian and Hamiltonian Graphs** ... 117
6.1 Introduction ... 117
6.2 Eulerian Graphs ... 117
6.3 Hamiltonian Graphs ... 122
6.3.1 Hamilton's "Around the World" Game ... 122
6.4 Pancyclic Graphs ... 130
6.5 Hamilton Cycles in Line Graphs ... 133
6.6 2-Factorable Graphs ... 138
6.7 Exercises ... 140
Notes ... 141

7 **Graph Colorings** ... 143
7.1 Introduction ... 143
7.2 Vertex Colorings ... 143
7.2.1 Applications of Graph Coloring ... 143
7.3 Critical Graphs ... 147
7.3.1 Brooks' Theorem ... 149
7.3.2 Other Coloring Parameters ... 151
7.3.3 b-Colorings ... 152
7.4 Homomorphisms and Colorings ... 153
7.4.1 Quotient Graphs ... 154

7.5 Triangle-Free Graphs 155
7.6 Edge Colorings of Graphs 159
7.6.1 The Timetable Problem 159
7.6.2 Vizing's Theorem 162
7.7 Snarks 167
7.8 Kirkman's Schoolgirl Problem 168
7.9 Chromatic Polynomials 170
Notes 173

8 Planarity 175
8.1 Introduction 175
8.2 Planar and Nonplanar Graphs 175
8.3 Euler Formula and Its Consequences 180
8.4 K_5 and $K_{3,3}$ are Nonplanar Graphs 184
8.5 Dual of a Plane Graph 186
8.6 The Four-Color Theorem and the Heawood Five-Color Theorem 189
8.7 Kuratowski's Theorem 191
8.8 Hamiltonian Plane Graphs 199
8.9 Tait Coloring 200
Notes 205

9 Triangulated Graphs 207
9.1 Introduction 207
9.2 Perfect Graphs 207
9.3 Triangulated Graphs 209
9.4 Interval Graphs 211
9.5 Bipartite Graph $B(G)$ of a Graph G 214
9.6 Circular Arc Graphs 215
9.7 Exercises 215
9.8 Phasing of Traffic Lights at a Road Junction 216
Notes 219

10 Domination in Graphs 221
10.1 Introduction 221
10.2 Domination in Graphs 221
10.3 Bounds for the Domination Number 224
10.4 Bound for the Size m in Terms of Order n and Domination Number $\gamma(G)$ 224
10.5 Independent Domination and Irredundance 227
10.6 Exercises 229
10.7 Vizing's Conjecture 229
10.8 Decomposable Graphs 234
10.9 Domination in Direct Products 237
Notes 238

11 Spectral Properties of Graphs 241
11.1 Introduction 241
11.2 The Spectrum of a Graph 241
11.3 Spectrum of the Complete Graph K_n 243
11.4 Spectrum of the Cycle C_n 243
11.4.1 Coefficients of the Characteristic Polynomial 244
11.5 The Spectra of Regular Graphs 244
11.5.1 The Spectrum of the Complement of a Regular Graph 245
11.5.2 Spectra of Line Graphs of Regular Graphs 246
11.6 Spectrum of the Complete Bipartite Graph $K_{p,q}$ 249
11.7 The Determinant of the Adjacency Matrix of a Graph 250
11.8 Spectra of Product Graphs 251
11.9 Cayley Graphs 253
11.9.1 Introduction 253
11.9.2 Unitary Cayley Graphs 254
11.9.3 Spectrum of the Cayley Graph X_n 255
11.10 Strongly Regular Graphs 255
11.11 Ramanujan Graphs 258
11.11.1 Why Are Ramanujan Graphs Important? 259
11.12 The Energy of a Graph 261
11.12.1 Introduction 261
11.12.2 Maximum Energy of k-Regular Graphs 262
11.12.3 Hyperenergetic Graphs 266
11.12.4 Energy of Cayley Graphs 267
11.13 Energy of the Mycielskian of a Regular Graph 269
11.13.1 An Application of Theorem 11.13.1 271
11.14 Exercises 272
Notes 273

List of Symbols 275

References 279

Index 287

Chapter 1
Basic Results

1.1 Introduction

Graphs serve as mathematical models to analyze many concrete real-world problems successfully. Certain problems in physics, chemistry, communication science, computer technology, genetics, psychology, sociology, and linguistics can be formulated as problems in graph theory. Also, many branches of mathematics, such as group theory, matrix theory, probability, and topology, have close connections with graph theory.

Some puzzles and several problems of a practical nature have been instrumental in the development of various topics in graph theory. The famous Königsberg bridge problem has been the inspiration for the development of Eulerian graph theory. The challenging Hamiltonian graph theory has been developed from the "Around the World" game of Sir William Hamilton. The theory of acyclic graphs was developed for solving problems of electrical networks, and the study of "trees" was developed for enumerating isomers of organic compounds. The well-known four-color problem formed the very basis for the development of planarity in graph theory and combinatorial topology. Problems of linear programming and operations research (such as maritime traffic problems) can be tackled by the theory of flows in networks. Kirkman's schoolgirl problem and scheduling problems are examples of problems that can be solved by graph colorings. The study of simplicial complexes can be associated with the study of graph theory. Many more such problems can be added to this list.

1.2 Basic Concepts

Consider a road network of a town consisting of streets and street intersections. Figure 1.1a represents the road network of a city. Figure 1.1b denotes the corresponding graph of this network, where the street intersections are represented by

R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory*,
Universitext, DOI 10.1007/978-1-4614-4529-6_1,

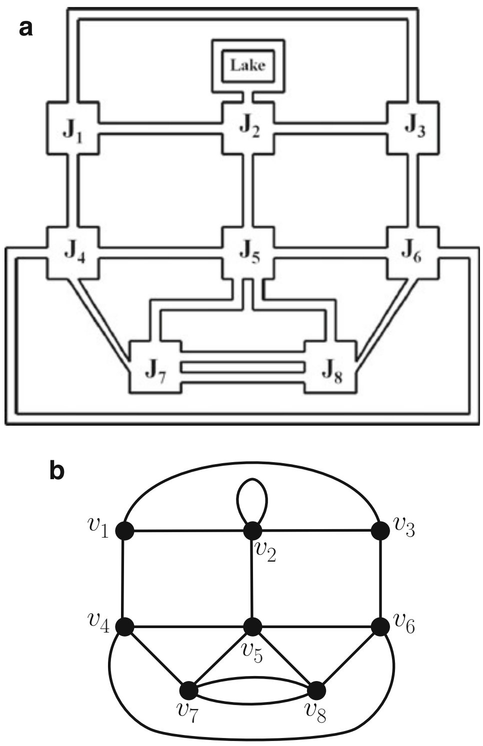


Fig. 1.1 (**a**) A road network and (**b**) the graph corresponding to the road network in (a)

points, and the street joining a pair of intersections is represented by an arc (not necessarily a straight line). The road network in Fig. 1.1 is a typical example of a graph in which intersections and streets are, respectively, the "vertices" and "edges" of the graph. (Note that in the road network in Fig. 1.1a, there are two streets joining the intersections J_7 and J_8, and there is a loop street starting and ending at J_2.)

We now present a formal definition of a graph.

Definition 1.2.1. A *graph* is an ordered triple $G = (V(G), E(G), I_G)$, where $V(G)$ is a nonempty set, $E(G)$ is a set disjoint from $V(G)$, and I_G is an "incidence" relation that associates with each element of $E(G)$ an unordered pair of elements (same or distinct) of $V(G)$. Elements of $V(G)$ are called the *vertices* (or *nodes* or *points*) of G, and elements of $E(G)$ are called the *edges* (or *lines*) of G. $V(G)$ and $E(G)$ are the *vertex set* and *edge set* of G, respectively. If, for the edge e of G, $I_G(e) = \{u, v\}$, we write $I_G(e) = uv$.

Example 1.2.2. If $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, and I_G is given by $I_G(e_1) = \{v_1, v_5\}$, $I_G(e_2) = \{v_2, v_3\}$, $I_G(e_3) = \{v_2, v_4\}$, $I_G(e_4) = \{v_2, v_5\}$, $I_G(e_5) = \{v_2, v_5\}$, $I_G(e_6) = \{v_3, v_3\}$, then $(V(G), E(G), I_G)$ is a graph.

Diagrammatic Representation of a Graph 1.2.3. Each graph can be represented by a diagram in the plane. In this diagram, each vertex of the graph is represented

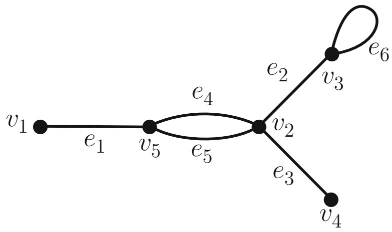


Fig. 1.2 Graph $(V(G), E(G), I_G)$ described in Example 1.2.2

by a point, with distinct vertices being represented by distinct points. Each edge is represented by a simple "Jordan" arc joining two (not necessarily distinct) vertices. The diagrammatic representation of a graph aids in visualizing many concepts related to graphs and the systems of which they are models. In a diagrammatic representation of a graph, it is possible that two edges intersect at a point that is not necessarily a vertex of the graph.

Definition 1.2.4. If $I_G(e) = \{u, v\}$, then the vertices u and v are called the *end vertices* or *ends* of the edge e. Each edge is said to join its ends; in this case, we say that e is *incident* with each one of its ends. Also, the vertices u and v are then *incident* with e. A set of two or more edges of a graph G is called a set of *multiple* or *parallel edges* if they have the same pair of distinct ends. If e is an edge with end vertices u and v, we write $e = uv$. An edge for which the two ends are the same is called a *loop* at the common vertex. A vertex u is a *neighbor* of v in G, if uv is an edge of G, and $u \neq v$. The set of all neighbors of v is the *open neighborhood* of v or the *neighbor set* of v, and is denoted by $N(v)$; the set $N[v] = N(v) \cup \{v\}$ is the *closed neighborhood* of v in G. When G needs to be made explicit, these open and closed neighborhoods are denoted by $N_G(v)$ and $N_G[v]$, respectively. Vertices u and v are *adjacent* to each other in G if and only if there is an edge of G with u and v as its ends. Two distinct edges e and f are said to be *adjacent* if and only if they have a common end vertex. A graph is *simple* if it has no loops and no multiple edges. Thus, for a simple graph G, the incidence function I_G is one-to-one. Hence, an edge of a simple graph is identified with the pair of its ends. A simple graph therefore may be considered as an ordered pair $(V(G), E(G))$, where $V(G)$ is a nonempty set and $E(G)$ is a set of unordered pairs of elements of $V(G)$ (each edge of the graph being identified with the pair of its ends).

Example 1.2.5. In the graph of Fig. 1.2, edge $e_3 = v_2v_4$, edges e_4 and e_5 form multiple edges, e_6 is a loop at v_3, $N(v_2) = \{v_3, v_4, v_5\}$, $N(v_3) = \{v_2\}$, $N[v_2] = \{v_2, v_3, v_4, v_5\}$, and $N[v_2] = N(v_2) \cup \{v_2\}$. Further, v_2 and v_5 are adjacent vertices and e_3 and e_4 are adjacent edges.

Definition 1.2.6. A graph is called *finite* if both $V(G)$ and $E(G)$ are finite. A graph that is not finite is called an *infinite* graph. Unless otherwise stated, all graphs considered in this text are finite. Throughout this book, we denote by $n(G)$ and $m(G)$ the number of vertices and edges of the graph G, respectively. The number $n(G)$ is called the *order* of G and $m(G)$ is the *size* of G. When explicit reference to

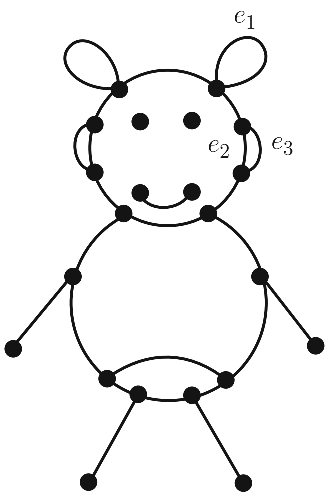


Fig. 1.3 A graph diagram; e_1 is a loop and $\{e_2, e_3\}$ is a set of multiple edges

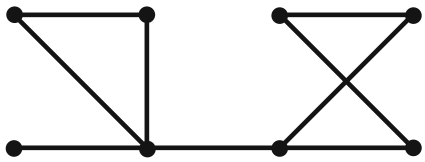


Fig. 1.4 A simple graph

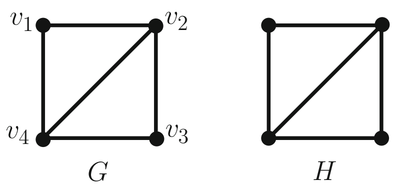


Fig. 1.5 A labeled graph G and an unlabeled graph H

the graph G is not needed, $V(G)$, $E(G)$, $n(G)$, and $m(G)$ will be denoted simply by V, E, n, and m, respectively.

Figure 1.3 is a graph with loops and multiple edges, while Fig. 1.4 represents a simple graph.

Remark 1.2.7. The representation of graphs on other surfaces such as a sphere, a torus, or a Möbius band could also be considered. Often a diagram of a graph is identified with the graph itself.

Definition 1.2.8. A graph is said to be *labeled* if its n vertices are distinguished from one another by labels such as $v_1, v_2, \ldots, v_n$ (see Fig. 1.5).

Note that there are three different labeled simple graphs on three vertices each having two edges, whereas there is only one unlabeled simple graph of the same order and size (see Fig. 1.6).

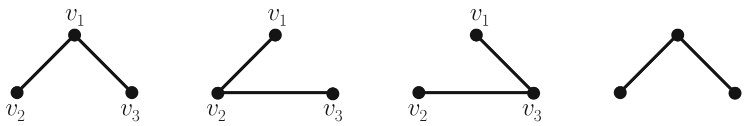


Fig. 1.6 Labeled and unlabeled simple graphs on three vertices

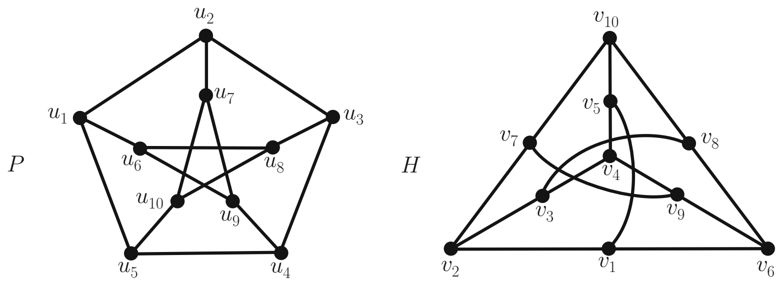


Fig. 1.7 Isomorphic graphs

Isomorphism of Graphs 1.2.9. A graph isomorphism, which we now define, is a concept similar to isomorphism in algebraic structures. Let $G = (V(G), E(G), I_G)$ and $H = (V(H), E(H), I_H)$ be two graphs. A *graph isomorphism* from G to H is a pair (ϕ, θ), where $\phi : V(G) \to V(H)$ and $\theta : E(G) \to E(H)$ are bijections with the property that $I_G(e) = \{u, v\}$ if and only if $I_H(\theta(e)) = \{\phi(u), \phi(v)\}$. If (ϕ, θ) is a graph isomorphism, the pair of inverse mappings (ϕ^{-1}, θ^{-1}) is also a graph isomorphism. Note that the bijection ϕ satisfies the condition that *u and v are end vertices of an edge e of G if and only if $\phi(u)$ and $\phi(v)$ are end vertices of the edge $\theta(e)$ in H*. It is clear that isomorphism is an equivalence relation on the set of all graphs. Isomorphism between graphs is denoted by the symbol $\simeq$ (as in algebraic structures).

Simple Graphs and Isomorphisms 1.2.10. If graphs G and H are simple, any bijection $\phi : V(G) \to V(H)$ such that u and v are adjacent in G if and only if $\phi(u)$ and $\phi(v)$ are adjacent in H induces a bijection $\theta : E(G) \to E(H)$ satisfying the condition that $I_G(e) = \{u, v\}$ if and only if $I_H(\theta(e)) = \{\phi(u), \phi(v)\}$. Hence, ϕ itself is referred to as an isomorphism in the case of simple graphs G and H. Thus, if G and H are simple graphs, an isomorphism from G to H is a bijection $\phi : V(G) \to V(H)$ such that u and v are adjacent in G if and only if $\phi(v)$ and $\phi(v)$ are adjacent in H. Figure 1.7 exhibits two isomorphic graphs P and H, where P is the well-known Petersen graph. We observe that P is a simple graph.

Exercise 2.1. Let G and H be simple graphs and let $\phi : V(G) \to V(H)$ be a bijection such that $uv \in E(G)$ implies that $\phi(u)\phi(v) \in E(H)$. Show by means of an example that ϕ need not be an isomorphism from G to H.

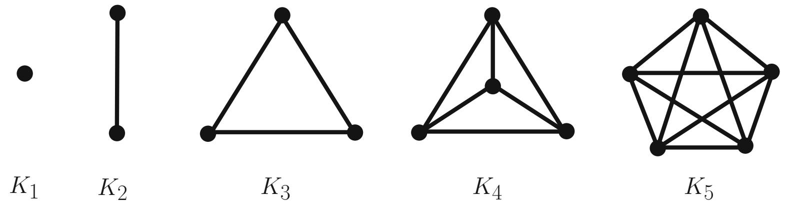
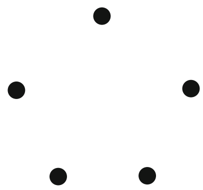


Fig. 1.8 Some complete graphs

Fig. 1.9 A totally disconnected graph on five vertices



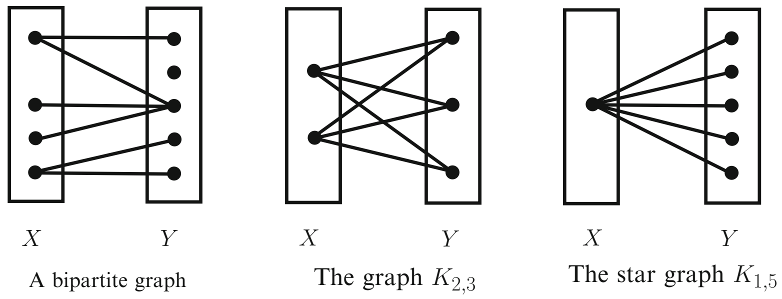


Fig. 1.10 Bipartite graphs

Definition 1.2.11. A simple graph G is said to be *complete* if every pair of distinct vertices of G are adjacent in G. Any two complete graphs each on a set of n vertices are isomorphic; each such graph is denoted by K_n (Fig. 1.8).

A simple graph with n vertices can have at most $\binom{n}{2} = \frac{n(n-1)}{2}$ edges. The complete graph K_n has the maximum number of edges among all simple graphs with n vertices. At the other extreme, a graph may possess no edge at all. Such a graph is called a *totally disconnected graph* (see Fig. 1.9). Thus, *for a simple graph G with n vertices, we have* $0 \leq m(G) \leq \frac{n(n-1)}{2}$.

Definition 1.2.12. A graph is *trivial* if its vertex set is a singleton and it contains no edges. A graph is *bipartite* if its vertex set can be partitioned into two nonempty subsets X and Y such that each edge of G has one end in X and the other in Y. The pair (X, Y) is called a *bipartition* of the bipartite graph. The bipartite graph G with bipartition (X, Y) is denoted by $G(X, Y)$. A simple bipartite graph $G(X, Y)$ is *complete* if each vertex of X is adjacent to all the vertices of Y. If $G(X, Y)$ is complete with $|X| = p$ and $|Y| = q$, then $G(X, Y)$ is denoted by $K_{p,q}$. A complete bipartite graph of the form $K_{1,q}$ is called a *star* (see Fig. 1.10).

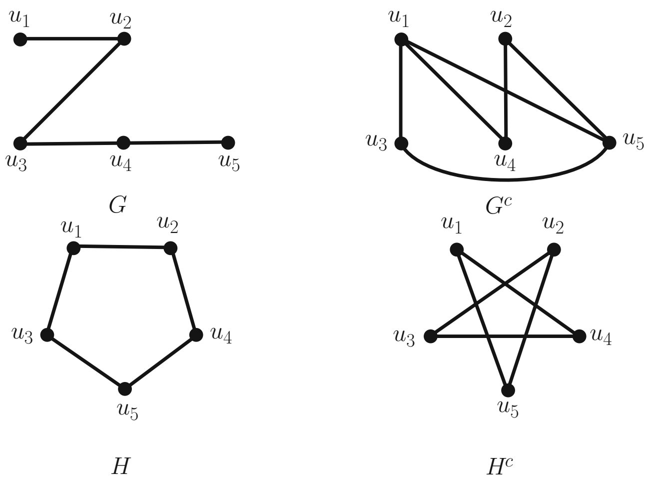
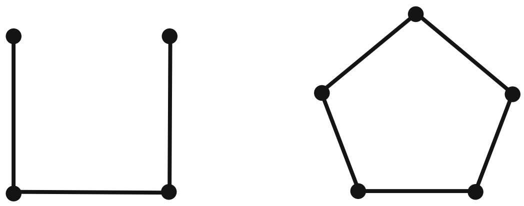


Fig. 1.11 Two simple graphs and their complements

Fig. 1.12 Self-complementary graphs



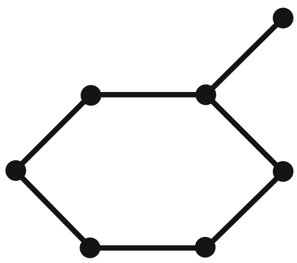
Definition 1.2.13. Let G be a simple graph. Then the *complement* G^c of G is defined by taking $V(G^c) = V(G)$ and making two vertices u and v adjacent in G^c if and only if they are nonadjacent in G (see Fig. 1.11). It is clear that G^c is also a simple graph and that $(G^c)^c = G$.

If $|V(G)| = n$, then clearly, $|E(G)| + |E(G^c)| = |E(K_n)| = \frac{n(n-1)}{2}$.

Definition 1.2.14. A simple graph G is called *self-complementary* if $G \simeq G^c$.

For example, the graphs shown in Fig. 1.12 are self-complementary.

Exercise 2.2. Find the complement of the following simple graph:



1.3 Subgraphs

Definition 1.3.1. A graph H is called a *subgraph* of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and I_H is the restriction of I_G to $E(H)$. If H is a subgraph of G, then G is said to be a *supergraph* of H. A subgraph H of a graph G is a *proper subgraph* of G if either $V(H) \neq V(G)$ or $E(H) \neq E(G)$. (Hence, when G is given, for any subgraph H of G, the incidence function is already determined so that H can be specified by its vertex and edge sets.) A subgraph H of G is said to be an *induced subgraph* of G if each edge of G having its ends in $V(H)$ is also an edge of H. A subgraph H of G is a *spanning subgraph* of G if $V(H) = V(G)$. The induced subgraph of G with vertex set $S \subseteq V(G)$ is called the *subgraph of* G *induced by* S and is denoted by $G[S]$. Let E' be a subset of E and let S denote the subset of V consisting of all the end vertices in G of edges in E'. Then the graph $(S, E', I_G|_{E'})$ is the *subgraph of* G *induced by the edge set* E' *of* G. It is denoted by $G[E']$ (see Fig. 1.13). Let u and v be vertices of a graph G. By $G + uv$, we mean the graph obtained by adding a new edge uv to G.

Definition 1.3.2. A *clique* of G is a complete subgraph of G. A clique of G is a *maximal clique* of G if it is not properly contained in another clique of G (see Fig. 1.13).

Definition 1.3.3. *Deletion of vertices and edges in a graph:* Let G be a graph, S a proper subset of the vertex set V, and E' a subset of E. The subgraph $G[V \backslash S]$ is said to be obtained from G by the *deletion* of S. This subgraph is denoted by $G - S$. If $S = \{v\}$, $G - S$ is simply denoted by $G - v$. The spanning subgraph of G with the edge set $E \backslash E'$ is the subgraph obtained from G by deleting the edge subset E'. This subgraph is denoted by $G - E'$. Whenever $E' = \{e\}$, $G - E'$ is

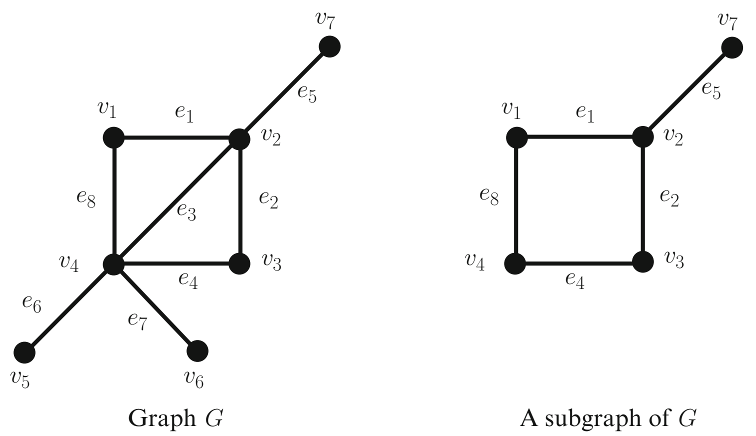


Fig. 1.13 Various subgraphs and cliques of G

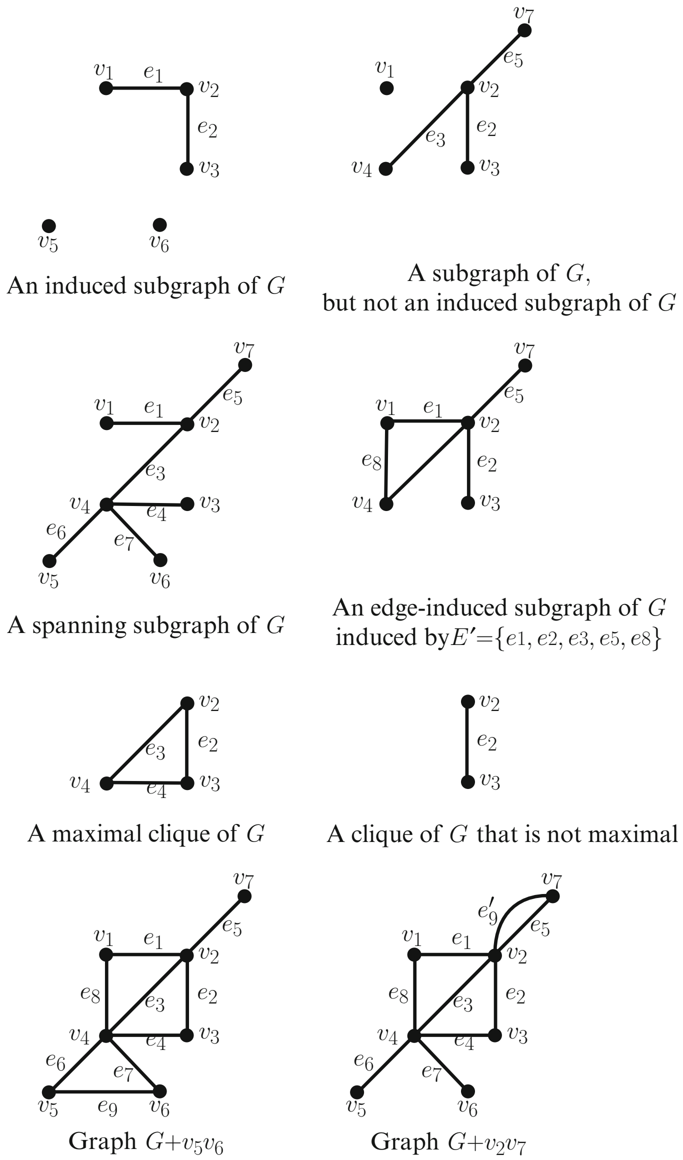


Fig. 1.13 (continued)

simply denoted by $G - e$. Note that when a vertex is deleted from G, all the edges incident to it are also deleted from G, whereas the deletion of an edge from G does not affect the vertices of G (see Fig. 1.14).

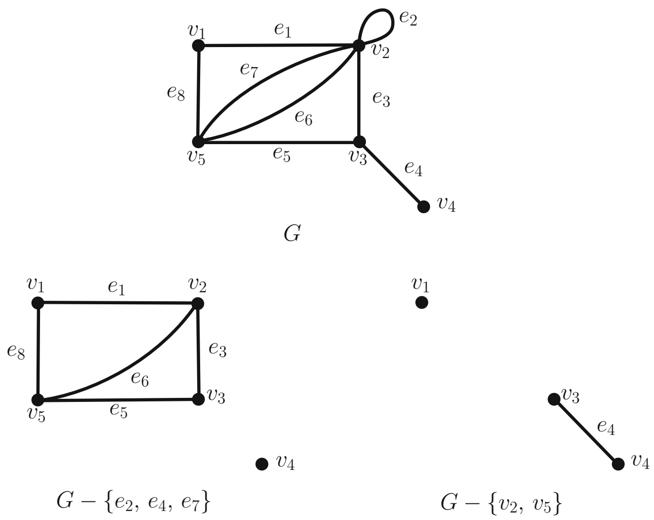


Fig. 1.14 Deletion of vertices and edges from G

1.4 Degrees of Vertices

Definition 1.4.1. Let G be a graph and $v \in V$. The number of edges incident at v in G is called the *degree* (or *valency*) of the vertex v in G and is denoted by $d_G(v)$, or simply $d(v)$ when G requires no explicit reference. A loop at v is to be counted twice in computing the degree of v. The minimum (respectively, maximum) of the degrees of the vertices of a graph G is denoted by $\delta(G)$ or δ (respectively, $\Delta(G)$ or Δ). A graph G is called k*-regular* if every vertex of G has degree k. A graph is said to be *regular* if it is k-regular for some nonnegative integer k. In particular, a 3-regular graph is called a *cubic graph.*

Definition 1.4.2. A spanning 1-regular subgraph of G is called a 1-*factor* or a *perfect matching* of G. For example, in the graph G of Fig. 1.15, each of the pairs $\{ab, cd\}$ and $\{ad, bc\}$ is a 1-factor of G.

Definition 1.4.3. A vertex of degree 0 is an *isolated vertex* of G. A vertex of degree 1 is called a *pendant vertex* of G, and the unique edge of G incident to such a vertex of G is a *pendant edge* of G. A sequence formed by the degrees of the vertices of G, when the vertices are taken in the same order, is called a *degree sequence* of G. It is customary to give this sequence in the nonincreasing or nondecreasing order, in which case the sequence is unique.

In the graph G of Fig. 1.16, the numbers within the parentheses indicate the degrees of the corresponding vertices. In G, v_7 is an isolated vertex, v_6 is a pendant vertex, and v_5v_6 is a pendant edge. The degree sequence of G is $(0, 1, 2, 2, 4, 4, 5)$.

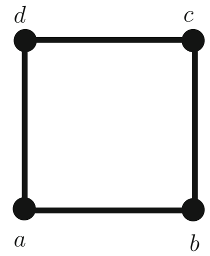


Fig. 1.15 Graph with 1-factors

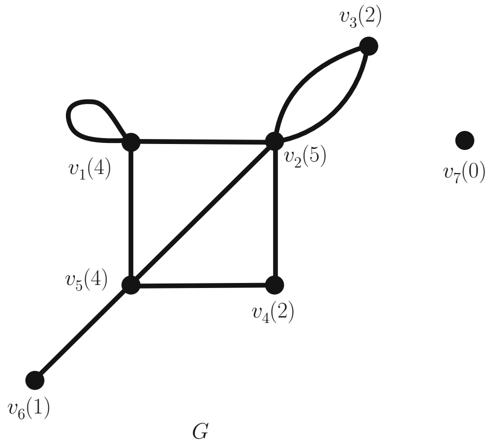


Fig. 1.16 Degrees of vertices of graph G

The very first theorem of graph theory was due to Leonhard Euler (1707–1783). This theorem connects the degrees of the vertices and the number of edges of a graph.

Theorem 1.4.4 (Euler). *The sum of the degrees of the vertices of a graph is equal to twice the number of its edges.*

Proof. If $e = uv$ is an edge of G, e is counted once while counting the degrees of each of u and v (even when $u = v$). Hence, each edge contributes 2 to the sum of the degrees of the vertices. Thus, the m edges of G contribute $2m$ to the degree sum. □

Remark 1.4.5. If $d = (d_1, d_2, \ldots, d_n)$ is the degree sequence of G, then the above theorem gives the equation $\sum_{i=1}^{n} d_i = 2m$, where n and m are the order and size of G, respectively.

Corollary 1.4.6. *In any graph G, the number of vertices of odd degree is even.*

Proof. Let V_1 and V_2 be the subsets of vertices of G with odd and even degrees, respectively. By Theorem 1.4.4,

$$2m(G) = \sum_{v \in V} d_G(v) = \sum_{v \in V_1} d_G(v) + \sum_{v \in V_2} d_G(v).$$