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From Real to Complex Analysis

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Preface

This book evolved from a series of lectures at the University of Sussex and is designed to provide an integrated course in real and complex analysis for undergraduates who have taken first steps in real analysis; the intention is to exhibit something of the interplay between these and other areas of mathematical study. The prerequisites are modest: it would be completely sufficient to have followed preliminary courses in real analysis (involving ε, δ ideas) and algebraic structures. There are many exercises, ranging from the elementary to the quite demanding. To establish notation and terminology, some prerequisites are reviewed in the appendices.

A persistent theme in the text is the search for a primitive. In the case of real analysis, the Riemann integral offers one route in this quest and, with an eye to complex analysis, the improper Riemann integral is an extension consonant with the demands of contour integrals.

Chapter 1 deals with the Riemann theory of integration on the real line using the simple and elegant approach due to Darboux that quickly leads to the basic properties of the integral together with means of evaluation and estimation. It also enables direct, elementary proofs to be given of the results that if f is Riemann-integrable, then (i) the set of its points of continuity is dense in the domain of f, and (ii) $g \circ f$ is Riemann-integrable if g is continuous. A characterization of the class of Riemann-integrable functions, from which these last two assertions follow, is postponed to the next chapter as it is technically more challenging. The Riemann integral is confined to bounded functions defined on closed bounded intervals and requires extension to cope with the demands of later chapters. To allow for some relaxation of these constraints, the improper Riemann integral is introduced. We indicate the limitations of the Riemann integral which led to the development of Lebesgue's integral (which itself would require slight extension for use in the later chapters), of which the former is a special case.

Metric spaces form the theme of Chap. 2; the earlier one provides a wealth of examples of such objects. Detailed coverage is given of the core properties of completeness, compactness, connectedness and simple connectedness: this last property is highlighted. While it has become more common in recent times to present such matters in the context of normed linear spaces, we believe it is important for the student to realize that linear structure is irrelevant to many of the results. Regarding completeness, Cantor's characterization is established as are

Banach's contraction mapping theorem and the Baire category theorem, the last leading to a proof of existence of a continuous, nowhere-differentiable function and also to the fact that the pointwise limit of a sequence of continuous real-valued functions on a complete metric space is continuous on a dense subset of that space. Compactness and connectedness are motivated in a variety of ways, the definitions chosen being intrinsic and applicable in more general contexts. Among the applications of compactness are differentiation under the integral sign, Peano's theorem on the existence of solutions of initial-value problems for certain nonlinear ordinary differential equations, and the characterization of Riemann-integrable functions as functions that are bounded and continuous almost everywhere. With the next chapter in mind, we conclude with the consideration of simply connected spaces. Various forms of homotopy are given especially detailed coverage, strenuous efforts being made to give complete proofs. We show that a metric space is simply connected if and only if it is path-connected and its fundamental group at any (and hence every) point of the space has only one element.

In Chap. 3, we reach our main goal, the theory of complex analysis, surely one of the most wonderful and fertile parts of mathematics. After some basic definitions and results, we deal with power series, branches of the argument and logarithm, continuous logarithms of continuous zero-free functions, the winding number for arbitrary paths in the plane and its invariance under free homotopy, and integrals over contours. Ample justification for the introduction of the winding number is provided by the demands of the proof of the Jordan curve theorem given later (for which the winding number is essential and the index is inadequate as it is undefined for general paths having no smoothness), but in addition we believe that there is a computational and pedagogical advantage in having this concept available. The homology version of Cauchy's theorem is derived by means of the elegant approach of Dixon [6]. Rudin [15] was one of the first to draw attention to the importance of Dixon's contribution and the organisation of complex analysis consequent upon it. Rather than appeal to an interchange of the order of integration, as Rudin does, we follow Dixon's original treatment and use differentiation under the integral sign. This leads to the residue theorem, from which flow such major theoretical results as Rouché's theorem and the open mapping and inverse function theorems; further, at a practical and technical level it is valuable in the evaluation of definite integrals. The penultimate section contains a result of exceptional aesthetic appeal which establishes, for connected open sets $G \subset \mathbf{C}$ (the space of all complex numbers) the equivalence of various statements of an analytic, algebraic and topological character. In particular, it shows that every function analytic on G has a primitive if and only if G is simply connected. In the course of the proof, such famous results as Montel's theorem and the Riemann mapping theorem are obtained. The final section reinforces the links between analysis and topology. Further study of topics introduced earlier, namely continuous logarithms of continuous zero-free functions and the winding number of a path, leads in a very natural way to a proof of the celebrated Jordan curve theorem. For this development of the theory, we acknowledge a major debt to the book [3] by Burckel. A beautiful result due to Borsuk concerning any compact set

$K \subset \mathbf{C}$ emerges in the course of the proof of the Jordan curve theorem: $\mathbf{C} \backslash K$ is connected if and only if every continuous function $f : K \to \mathbf{C} \backslash \{0\}$ has a continuous logarithm.

Our exposition covers aspects of classical analysis due to the efforts of generations of mathematicians. There is no claim to originality save for the selection and presentation of material. We have been greatly influenced by the scholarly and inspirational books by Burckel [3], Remmert [13] and Rudin [15], and hope readers of the present book will go on to consult these more advanced and wider-ranging works.

It is a pleasure to acknowledge our great indebtedness to Dorothee Haroske for her immense help and patience. Finally, we express our appreciation to Joerg Sixt and his staff at the London Office of Springer-Verlag for constant encouragement and advice.

Contents

Chapter 1
The Riemann Integral

In this chapter we give an account of the Riemann integral for real-valued functions defined on intervals of the real line. This integral is of historic interest, has considerable intuitive appeal and possesses great practical value. For economy of presentation we use the approach of Darboux rather than that originally employed by Riemann.

Hidden from immediate view but at the heart of the chapter lies the sense in which integration is the inverse of differentiation. For the class of continuous functions the Riemann integral provides an affirmative answer to the question "Given $f : [a, b] \to \mathbf{R}$, where a and b are real and $a < b$, does there exist $F : [a, b] \to \mathbf{R}$ such that $F' = f$?" With somewhat greater effort, development of the Lebesgue integral would allow us to enlarge this class. However, for the topics covered in this text the answer provided suffices; in particular, it is entirely adequate in the resolution of an analogous question asked in the context of complex analysis, a question which is the focus of our final chapter.

1.1 Basic Definitions and Results

Definition 1.1.1 Let a and b be real numbers, with $a < b$. Any finite set of points $x_0, x_1, \ldots, x_n$ with $a = x_0 < x_1 < \ldots < x_n = b$ is called a **partition** of $[a, b]$ and will often be denoted by P; we put $\Delta x_j = x_j - x_{j-1}$ $(j = 1, \ldots, n)$ and call $w(P) := \max \left\{\Delta x_j : j = 1, \ldots, n\right\}$ the **width** of P. The family of all partitions of $[a, b]$ is denoted by $\mathscr{P}[a, b]$, or simply by $\mathscr{P}$ if no ambiguity is possible. Let $\mathscr{B}[a, b]$ (or simply $\mathscr{B}$) be the family of all bounded functions $f : [a, b] \to \mathbf{R}$; given any $f \in \mathscr{B}$ and any $P \in \mathscr{P}$, put

$$M_j = \sup \left\{ f(x) : x_{j-1} \leq x \leq x_j \right\}, \; m_j = \inf \left\{ f(x) : x_{j-1} \leq x \leq x_j \right\}$$

for $j = 1, \ldots, n$ and call

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$$U(P, f) := \sum_{j=1}^{n} M_j \Delta x_j, \; L(P, f) := \sum_{j=1}^{n} m_j \Delta x_j$$

the **upper** and **lower sums of** f **with respect to** P, respectively.

Note that $U(P, f)$ is the sum of the signed areas of n rectangles, the jth of which has base Δx_j and height M_j; $L(P, f)$ is the same except that the jth rectangle has height m_j. These quantities are familiar to anyone who has tried to estimate the area of the set of points lying between the curve $y = f(x)$ and the lines $x = a$, $x = b$ and $y = 0$ by drawing the graph of f on squared paper: $U(P, f)$ arises from consistent over-estimation of the area by rectangles above the graph, while $L(P, f)$ comes from a corresponding lower estimation by rectangles below the graph.

Example 1.1.2

(i) Let $f : [a, b] \to \mathbf{R}$ be monotonic increasing and let

$P = \{a = x_0, x_1, \ldots, x_n = b\}$ be a partition of $[a, b]$. Then

$$U(P, f) := \sum_{j=1}^{n} f(x_j)\Delta x_j, \; L(P, f) := \sum_{j=1}^{n} f(x_{j-1})\Delta x_j.$$

(ii) Let $f : [a, b] \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} 1, & x \text{ rational}, \\ -1, & x \text{ irrational}. \end{cases}$$

Then given any partition P of $[a, b]$, $M_j = 1$ and $m_j = -1$ $(j = 1, \ldots, n)$ since each interval $[x_{j-1}, x_j]$ contains both rational and irrational points. Hence

$$U(P, f) = b - a, \; L(P, f) = -(b - a).$$

Now let $f \in \mathscr{B}[a, b]$; that is, let f be a bounded, real-valued function on $[a, b]$. Since f is bounded, there are numbers $m, M \in \mathbf{R}$ such that for all $x \in [a, b]$, $m \leq f(x) \leq M$. Hence for all $P \in \mathscr{P}[a, b]$,

$$m(b-a) = \sum_{j=1}^{n} m\Delta x_j \leq \sum_{j=1}^{n} m_j \Delta x_j \leq \sum_{j=1}^{n} M_j \Delta x_j \leq \sum_{j=1}^{n} M\Delta x_j = M(b-a);$$

that is,

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a).$$

Thus $\{U(P, f) : P \in \mathscr{P}\}$ and $\{L(P, f) : P \in \mathscr{P}\}$ are bounded sets of real numbers; consequently they have a finite infimum and supremum.

Definition 1.1.3 Let $f \in \mathscr{B}[a, b]$. The **upper** and **lower integrals** of f **over** $[a, b]$ are

$$\overline{\int_a^b} f := \inf\{U(P, f) : P \in \mathscr{P}\}, \quad \underline{\int_a^b} f := \sup\{L(P, f) : P \in \mathscr{P}\},$$

respectively. If these upper and lower integrals are equal, we say that f is **Riemann-integrable over** $[a, b]$ and write

$$\int_a^b f = \overline{\int_a^b} f \; (= \underline{\int_a^b} f);$$

$\int_a^b f$, often written $\int_a^b f(x)dx$, is called the **Riemann integral of** f **over** $[a, b]$. The family of all functions which are Riemann-integrable over $[a, b]$ is denoted by $\mathscr{R}[a, b]$, or simply by $\mathscr{R}$.

Example 1.1.4

(i) Let $c \in \mathbf{R}$ and let $f : [a, b] \to \mathbf{R}$ be defined by $f(x) = c$ for all $x \in [a, b]$. Then for all $P \in \mathscr{P}[a, b]$, $U(P, f) = L(P, f) = c(b - a)$; hence $\overline{\int_a^b} f = \underline{\int_a^b} f = c(b - a)$ and so $f \in \mathscr{R}[a, b]$ with $\int_a^b f = c(b - a)$.

(ii) For the function f of Example 1.1.2 (ii), evidently $\overline{\int_a^b} f = b - a$ and $\underline{\int_a^b} f := -(b - a)$, so that $f \notin \mathscr{R}[a, b]$. However, example (i) above shows that despite this, $|f| \in \mathscr{R}[a, b]$.

We now proceed to investigate the family $\mathscr{R}[a, b]$ and to develop various properties of the integral.

Definition 1.1.5 Given any two partitions P, Q of $[a, b]$, Q is called a **refinement of** P if $P \subset Q$; that is, if every point of P is a point of Q. If $P_1, P_2 \in \mathscr{P}[a, b]$, then $Q := P_1 \cup P_2$ is called the **common refinement** of P_1 and P_2.

Lemma 1.1.6 *Let $f \in \mathscr{B}[a, b]$, let $K \in \mathbf{R}$ be such that $|f(x)| \le K$ whenever $x \in [a, b]$, and let $P \in \mathscr{P}[a, b]$. If $Q \in \mathscr{P}[a, b]$ and Q is a refinement of P with exactly k points in addition to those of P, then*

$$\text{(i)} \quad 0 \le U(P, f) - U(Q, f) \le 2kKw(P)$$

and

$$\text{(ii)} \quad 0 \le L(Q, f) - L(P, f) \le 2kKw(P).$$

Proof It suffices to prove (i), since (ii) follows on the observation that

$$U(P, -f) = -L(P, f)$$

(see Exercise 1.1.10/3).

The proof of (i) when $k = 1$ is almost trivial. Let $P = \{x_0, x_1, \ldots, x_n\}$, let $x_* \in (x_{j-1}, x_j)$ for some $j \in \{1, 2, \ldots, n\}$ and let $Q = P \cup \{x_*\}$. Let

$$M_j^* = \sup\left\{f(x) : x_{j-1} \le x \le x_*\right\}, M_j^{**} = \sup\left\{f(x) : x_* \le x \le x_j\right\};$$

evidently $M_j^*, M_j^{**} \le M_j$. Now

$$U(P, f) - U(Q, f) = (M_j - M_j^*)(x_* - x_{j-1}) + (M_j - M_j^{**})(x_j - x_*),$$

since the other terms of the upper sums cancel. Hence

$$0 \le U(P, f) - U(Q, f) \le 2Kw(P).$$

Now suppose (i) is false for some $k \in \mathbf{N}$. Then there is a least $k_0 \in \mathbf{N}$, necessarily greater than 1, and an associated $Q_0 \in \mathscr{P}[a, b]$ with precisely k_0 points in addition to those of P, such that

$$U(P, f) - U(Q_0, f) \notin [0, 2k_0Kw(P)]. \tag{1.1.1}$$

Delete one point from Q_0 which does not lie in P and let Q_1 be the resulting partition of $[a, b]$. By what has already been proved,

$$0 \le U(Q_1, f) - U(Q_0, f) \le 2Kw(Q_1) \le 2Kw(P).$$

Further, since (i) holds for $k = k_0 - 1$,

$$0 \le U(P, f) - U(Q_1, f) \le 2(k_0 - 1)Kw(P).$$

Addition shows that

$$0 \le U(P, f) - U(Q_0, f) \le 2k_0Kw(P),$$

which contradicts (1.1.1) and proves that (i) is true for all k. □

Lemma 1.1.6 is very useful: it shows that the upper and lower sums are decreasing and increasing respectively on refinement of a partition, and enables the changes in these sums on refinement to be estimated. It plays a key rôle in the proof of the following theorem due to Darboux, a theorem which is a cornerstone of the theory as we shall develop it.

Theorem 1.1.7 *Let $f \in \mathscr{B}[a, b]$ and let (P_n) be a sequence in $\mathscr{P}[a, b]$ such that $\lim_{n\to\infty} w(P_n) = 0$. Then*

$$\lim_{n\to\infty} U(P_n, f) = \overline{\int_a^b} f, \quad \lim_{n\to\infty} L(P_n, f) = \underline{\int_a^b} f.$$

In particular, $f \in \mathscr{R}[a, b]$ if, and only if, $\lim_{n\to\infty} \{U(P_n, f) - L(P_n, f)\} = 0$.

Proof Let $K \in \mathbf{R}$ be such that $|f(x)| < K$ for all $x \in [a, b]$, and let $\varepsilon > 0$. By definition of the upper integral, there exists $Q \in \mathscr{P}[a, b]$ such that

$$U(Q, f) < \overline{\int_a^b} f + \varepsilon/2.$$

Let Q have exactly k points. For each $n \in \mathbf{N}$, $P_n \cup Q$ is a refinement of P_n with at most k additional points; thus by Lemma 1.1.6,

$$\begin{aligned} U(P_n, f) &\leq 2kKw(P_n) + U(P_n \cup Q, f) \\ &\leq 2kKw(P_n) + U(Q, f) \\ &\leq 2kKw(P_n) + \overline{\int_a^b} f + \varepsilon/2. \end{aligned}$$

Now, by hypothesis, there exists $N \in \mathbf{N}$ such that $w(P_n) < \varepsilon/(4kK)$ whenever $n \geq N$. It follows that, for $n \geq N$, we have

$$0 \leq U(P_n, f) - \overline{\int_a^b} f < \varepsilon.$$

Hence $\lim_{n\to\infty} U(P_n, f) = \overline{\int_a^b} f$. Since $\overline{\int_a^b}(-f) = -\underline{\int_a^b} f$, the rest follows directly. □

Corollary 1.1.8 *For all $f \in \mathscr{B}[a, b]$, $\underline{\int_a^b} f \leq \overline{\int_a^b} f$.*

Proof Let (P_n) be a sequence in $\mathscr{P}[a, b]$ such that $w(P_n) \to 0$. By Theorem 1.1.7,

$$\underline{\int_a^b} f = \lim_{n\to\infty} L(P_n, f) \leq \lim_{n\to\infty} U(P_n, f) = \overline{\int_a^b} f.$$

□

The power of Theorem 1.1.7 is considerable. We use it in the next three sections to exhibit large classes of integrable functions, to give a rapid exposition of the basic properties of the integrals defined above, and to provide, at least in principle, a technique for their evaluation. Before engaging in such matters, however, we prove an equivalent version of it.

Theorem 1.1.9 *Let $f \in \mathscr{B}[a, b]$. Then*

$$\lim_{w(P)\to 0} U(P, f) = \overline{\int_a^b} f;$$

that is, given any $\varepsilon > 0$, there exists $\delta > 0$ such that $0 \leq U(P, f) - \overline{\int_a^b} f < \varepsilon$ if $P \in \mathscr{P}[a, b]$ and $w(P) < \delta$. Moreover,

$$\lim_{w(P)\to 0} L(P, f) = \underline{\int_a^b} f.$$

Proof To obtain a contradiction suppose that, for some $f \in \mathscr{B}[a, b]$,

$$\lim_{w(P)\to 0} U(P, f) \neq \overline{\int_a^b} f.$$

Then an $\varepsilon > 0$ exists such that, for each $n \in \mathbf{N}$, there is a $P_n \in \mathscr{P}[a, b]$ with the properties

$$\text{(i) } w(P_n) < 1/n, \text{ and (ii) } U(P_n, f) > \overline{\int_a^b} f + \varepsilon.$$

The first property shows that the sequence $(w(P_n))$ converges to zero; the second, in conjunction with Theorem 1.1.7, that

$$\overline{\int_a^b} f = \lim_{n\to\infty} U(P_n, f) \geq \overline{\int_a^b} f + \varepsilon,$$

which is impossible for a positive ε. □

Exercise 1.1.10

1. Let A and B be non-empty, bounded subsets of $\mathbf{R}$ and let $\lambda \in \mathbf{R}$. Put

$$A + B = \{a + b : a \in A, b \in B\} \text{ and } \lambda A = \{\lambda a : a \in A\}.$$

Show that

(i) $\sup(A + B) = \sup A + \sup B$,

(ii) $\inf(A + B) = \inf A + \inf B$,

(iii) $\sup(\lambda A) = \begin{cases} \lambda \sup A & \text{if } \lambda \geq 0, \\ \lambda \inf A & \text{if } \lambda < 0, \end{cases}$

(iv) $\inf(\lambda A) = \begin{cases} \lambda \inf A & \text{if } \lambda \geq 0, \\ \lambda \sup A & \text{if } \lambda < 0. \end{cases}$

2. Let A be a non-empty subset of a set X and let $f : X \to \mathbf{R}$ be bounded. The **oscillation of** f **over** A, $\mathrm{osc}(f; A)$, is defined to be

$$\sup\{|f(x) - f(y)| : x, y \in A\}.$$

Prove that

$$\mathrm{osc}(f; A) = \sup\{f(x) : x \in A\} - \inf\{f(x) : x \in A\}.$$

3. Let $f : [a, b] \to \mathbf{R}$ be bounded and let $P \in \mathscr{P}[a, b]$. Prove that

$$U(P, -f) = -L(P, f) \text{ and } L(P, -f) = -U(P, f).$$

4. Using merely the definition of integrability, show that the function f from $[0, 1]$ to $\mathbf{R}$ defined by $f(t) = t^2$ $(0 \le t \le 1)$ is Riemann-integrable over $[0, 1]$ and that

$$\int_0^1 f = 1/3.$$

[Show that (i) if $P \in \mathscr{P}[0, 1]$ then $U(P, f) \ge 1/3 \ge L(P, f)$; (ii) if P_n is that partition of $[0, 1]$ which divides it into n subintervals of equal length, then

$$U(P_n, f) = (n + 1)(2n + 1)/6n^2 \text{ and } L(P_n, f) = (n - 1)(2n - 1)/6n^2.]$$

5. Suppose $a < b$ and let $f : [a, b] \to \mathbf{R}$ be bounded and such that $f(t) > 0$ for all $t \in [a, b]$. Prove that $\underline{\int_a^b} f > 0$.
[A subinterval $[c, d]$ of $[a, b]$, with $c < d$, and an $\varepsilon > 0$ exist such that $\sup\big\{f(t) : \alpha \le t \le \beta\big\} \ge \varepsilon$ whenever $c \le \alpha < \beta \le d$.]
6. (Riemann's criterion for integrability.) Let $f \in \mathscr{B}[a, b]$. Then $f \in \mathscr{R}[a, b]$ if, and only if, given any $\varepsilon > 0$, there exists $P \in \mathscr{P}[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon.$$

7. Let $f : [a, b] \to \mathbf{R}$. Prove that $f \in \mathscr{R}[a, b]$ if, and only if, there exists a real number A $\left(= \int_a^b f\right)$ with the following property: for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| A - \sum_{j=1}^{n} f(\xi_j)\Delta x_j \right| < \varepsilon$$

whenever $P = \{x_0, x_1, \ldots, x_n\} \in \mathscr{P}[a, b]$, $w(P) < \delta$ and $\xi_j \in [x_{j-1}, x_j]$ for each $j \in \{1, 2, \ldots, n\}$.
8. Let $f : [0, 1] \to \mathbf{R}$ be defined by $f(x) = \sqrt{x}$ $(0 \le x \le 1)$. Let

$$P_n = \left\{0, \left(\frac{1}{n}\right)^2, \left(\frac{2}{n}\right)^2, \ldots, \left(\frac{n-1}{n}\right)^2, 1\right\}.$$

Calculate $w(P_n)$ and show that $\lim_{n\to\infty} w(P_n) = 0$. Determine $L(f, P_n)$ and $U(f, P_n)$, and show that f is Riemann-integrable over $[0, 1]$ and that

$$\int_0^1 \sqrt{x}dx = 2/3.$$

1.2 Classes of Integrable Functions

Clearly, the utility of any theory of integration depends on commonly encountered types of function having an integral under that theory. Continuous real-valued functions defined on closed bounded intervals are of such a type. Beginning with some preliminary discussion of continuity, we show that if $f : [a, b] \to \mathbf{R}$ is continuous then $f \in \mathscr{R}[a, b]$.

Definition 1.2.1 Let I be a non-empty interval in $\mathbf{R}$ and let $f : I \to \mathbf{R}$. The function f is said to be **continuous at** $x_0 \in I$ if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in I$ and $|x_0 - x| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$; it is said to be **continuous on** I if it is continuous at each point of I. We say that f is **uniformly continuous on** I if given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in I$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Note that the essential difference between continuity and uniform continuity on I is that while for uniform continuity the number δ depends only on ε, in the case of continuity δ depends on ε *and on* x_0: there may be no single δ which will achieve the desired smallness of $|f(x) - f(x_0)|$ *for all* $x_0 \in I$. Although uniform continuity on I evidently implies continuity on I, in general the converse is false. The following examples may help to understand the distinction between these two forms of continuity.

Example 1.2.2

(i) Let $I = [0, 1]$ and suppose that $f(x) = x^2$ for all $x \in I$. Then f is uniformly continuous on I, for if $\varepsilon > 0$ then

$$|f(x) - f(y)| = |(x + y)(x - y)| \le 2\,|x - y| < \varepsilon$$

if $x, y \in I$ and $|x - y| < \varepsilon/2$. Hence we may take $\delta = \varepsilon/2$.

(ii) Let $I = [0, \infty)$ and again suppose that $f(x) = x^2$ for all $x \in I$. Then f is continuous on I: to see this let $x_0 \in I$ and $\varepsilon > 0$. Given any $\delta > 0$ and any $x \in I$ such that $|x - x_0| < \delta$ it follows that $x + x_0 < 2x_0 + \delta$, and hence

$$|f(x) - f(x_0)| = |(x - x_0)(x + x_0)| < \delta(2x_0 + \delta).$$

The choice of any positive number δ less than $\eta := (x_0^2 + \varepsilon)^{1/2} - x_0$, say $\delta = \eta/2$, now shows that $|f(x) - f(x_0)| < \varepsilon$ if $|x - x_0| < \delta$, and the continuity of f on I is established. Note the dependence of δ on x_0. However, f is not uniformly continuous on I: for if it were, then given any $\varepsilon > 0$, there would exist $\delta > 0$ such that if $x, y \in I$ and $|x - y| < \delta$, then $\left|x^2 - y^2\right| < \varepsilon$. But given *any* $\delta > 0$, if we choose $n \in \mathbf{N}$, $x = n$ and $y = n + \frac{1}{2}\delta$, then $|x - y| < \delta$ but $\left|x^2 - y^2\right| = \left|(n + \frac{1}{2}\delta)^2 - n^2\right| = \delta n + \frac{1}{4}\delta^2$, which can be made arbitrarily large by choosing n large enough.

(iii) Let $I = (0, 1)$ and suppose that $f(x) = 1/x$ for all $x \in I$. This function is continuous on I: for given any $x_0 \in I$ and any $\varepsilon > 0$, we see that if $x \in I$ and $|x - x_0| < \delta < x_0$,

$$|f(x) - f(x_0)| < \delta/\{x_0(x - x_0 + x_0)\} < \delta/\{x_0(x_0 - \delta)\};$$

thus to obtain $|f(x) - f(x_0)| < \varepsilon$ we simply choose $\delta < \varepsilon x_0^2/(1+\varepsilon x_0)$. It is not possible to choose δ independent of x_0; that is, f is not uniformly continuous on I. To see this, it is merely necessary to observe that $\frac{1}{n} - \frac{1}{n+1} \to 0$ as $n \to \infty$, while $\left|f(\frac{1}{n}) - f(\frac{1}{n+1})\right| = 1$ for all $n \in \mathbf{N}$.

If I is a closed, bounded interval $[a, b]$ the distinction between continuity and uniform continuity on I disappears. To establish this it is convenient to appeal to the famous Bolzano-Weierstrass theorem: every bounded sequence of real numbers has a convergent subsequence. A proof of this theorem is given in Theorem A.4.13 of the Appendix.

Theorem 1.2.3 *Let $a, b \in \mathbf{R}$, with $a < b$. A function $f : [a, b] \to \mathbf{R}$ is continuous on $[a, b]$ if, and only if, it is uniformly continuous on $[a, b]$.*

Proof Suppose first that f is continuous, but not uniformly continuous on $[a, b]$. Then there exists $\varepsilon > 0$ such that given any $n \in \mathbf{N}$, there are points $x_n, y_n \in [a, b]$ with $|x_n - y_n| < 1/n$ and $|f(x_n) - f(y_n)| \geq \varepsilon$. The sequence (x_n) is bounded and so, by the Bolzano-Weierstrass theorem, has a convergent subsequence $(x_{m(n)})$ with $\lim_{n\to\infty} x_{m(n)} = x$, say; clearly $\lim_{n\to\infty} y_{m(n)} = x$. In view of the continuity of f,

$$\lim_{n\to\infty} \left\{f(x_{m(n)}) - f(y_{m(n)})\right\} = f(x) - f(x) = 0.$$

But, for all n, $\left|f(x_{m(n)}) - f(y_{m(n)}\right| \geq \varepsilon$. Thus

$$\lim_{n\to\infty} \left|f(x_{m(n)}) - f(y_{m(n)})\right| \geq \varepsilon,$$

which gives a contradiction. Hence f is uniformly continuous on $[a, b]$. The converse is obvious. □

An important result, given next, is an immediate consequence of Theorem 1.2.3. (The reader should also know a direct proof of it, one which bypasses the notion of uniform continuity. See Exercise 1.2.14/1.)

Theorem 1.2.4 *Let $a, b \in \mathbf{R}$, with $a < b$, and let $f : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$. Then f is bounded.*

Proof Since f is uniformly continuous on $[a, b]$, there exists $\delta > 0$ such that $|f(x) - f(y)| < 1$ if $x, y \in [a, b]$ and $|x - y| < \delta$. Choose $P = \{x_0, x_1, \ldots, x_n\} \in \mathscr{P}[a, b]$ such that $w(P) < \delta$. Then

$$\sup_{x \in [a,b]} |f(x)| \leq 1 + \max_{1 \leq i \leq n} |f(x_i)| .$$

The proof is complete. □

Armed with this equipment we now return to integration.

Theorem 1.2.5 *Let $a, b \in \mathbf{R}$ with $a < b$, and let $f : [a, b] \to \mathbf{R}$. Then:*

(i) *if f is monotone, $f \in \mathscr{R}[a, b]$;*
(ii) *if f is continuous on $[a, b]$, $f \in \mathscr{R}[a, b]$.*

Proof For $n \in \mathbf{N}$, let $P_n = \{x_0, x_1, \ldots, x_n\} \in \mathscr{P}[a, b]$ be such that $\Delta x_j = (b - a)/n$ $(j = 1, 2, \ldots, n)$; plainly the sequence (P_n) has the property that $\lim_{n\to\infty} w(P_n) = 0$.
(i) Suppose f is increasing on $[a, b]$ (otherwise consider $-f$). Then $M_j = f(x_j)$, $m_j = f(x_{j-1})$ $(j = 1, 2, \ldots, n)$ and

$$\begin{aligned} U(P_n, f) - L(P_n, f) &= (b - a)n^{-1} \sum_{j=1}^{n} \left\{ f(x_j) - f(x_{j-1}) \right\} \\ &= (b - a)n^{-1} \{ f(b) - f(a) \} \to 0 \end{aligned}$$

as $n \to \infty$. Thus by Theorem 1.1.7, $f \in \mathscr{R}[a, b]$.
(ii) By Theorem 1.2.4, $f \in \mathscr{B}[a, b]$. Let $\varepsilon > 0$. By Theorem 1.2.3, f is uniformly continuous on $[a, b]$; hence there exists $\delta > 0$ such that if $s, t \in [a, b]$ and $|s - t| < \delta$, then we have $|f(s) - f(t)| < \varepsilon$. Suppose $n \in \mathbf{N}$ is such that $w(P_n) < \delta$. Then for each $j \in \{1, 2, \ldots, n\}$,

$$M_j - m_j = \sup \left\{ |f(s) - f(t)| : s, t \in [x_{j-1}, x_j] \right\}$$

(see Exercise 1.1.10/2), and hence

$$U(P_n, f) - L(P_n, f) = \sum_{j=1}^{n} (M_j - m_j) \Delta x_j \leq \varepsilon (b - a).$$

Since $w(P_n) < \delta$ for all save finitely many n, Theorem 1.1.7 and passage to the limit as $n \to \infty$ show that

$$0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le \varepsilon(b-a).$$

The final inequality being valid for all positive ε, it follows that $\overline{\int_a^b} f = \underline{\int_a^b} f$, that is, $f \in \mathscr{R}[a,b]$. □

Note that there are Riemann-integrable functions which are neither continuous nor monotone: see Exercise 1.2.14/4 for one such example. This fact notwithstanding, although a Riemann-integrable function need not be continuous, it must have a point of continuity, indeed, infinitely many such. The next two lemmas are a preparation to prove this assertion.

Lemma 1.2.6 *Let $f \in \mathscr{R}[a,b]$ and let $a < c < d < b$. Then $f \in \mathscr{R}[c,d]$; more precisely, the restriction of f to $[c,d]$ belongs to $\mathscr{R}[c,d]$.*

Proof Let (P_n) be a sequence of partitions of $[a,b]$ such that $\{c,d\} \subset P_n$ $(n \in \mathbf{N})$ and $w(P_n) \to 0$. Let $Q_n = P_n \cap [c,d]$ $(n \in \mathbf{N})$. Then each $Q_n \in \mathscr{P}[c,d]$ and $w(Q_n) \to 0$. Since

$$U(Q_n, f) - L(Q_n, f) \le U(P_n, f) - L(P_n, f) \; (n \in \mathbf{N})$$

and $f \in \mathscr{R}[a,b]$, it follows that

$$0 \le \overline{\int_c^d} f - \underline{\int_c^d} f \le \overline{\int_a^b} f - \underline{\int_a^b} f = 0.$$

Thus $f \in \mathscr{R}[c,d]$. □

Lemma 1.2.7 *Let $f \in \mathscr{R}[a,b]$ and $v > 0$. Then there exists a closed interval $[c,d] \subset [a,b]$ such that*

(i) $a < c < d < b$,
(ii) $d - c < v$,
(iii) $\operatorname{osc}(f; [c,d]) < v$.

Proof Since $f \in \mathscr{R}[a,b]$, by Exercise 1.1.10/6, there is a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ such that

$$U(P,f) - L(P,f) = \sum_{j=1}^{n} (M_j - m_j)\Delta x_j < v(b-a).$$

Appealing to Exercise 1.1.10/2, it follows that, for some i $(1 \le i \le n)$,

$$\operatorname{osc}(f;[x_{i-1},x_i]) = M_i - m_i = \min_{1\le j\le n}(M_j - m_j) < v.$$

Finally, any choice of closed interval $[c, d]$ such that $x_{i-1} < c < d < x_i$ and $d - c < v$ has the properties required. □

Theorem 1.2.8 *Let $f \in \mathscr{R}[a, b]$. Then there is a real number c such that $a < c < b$ and f is continuous at c.*

Proof Since $f \in \mathscr{R}[a, b]$, by Lemma 1.2.7, there is a closed interval $[a_1, b_1] \subset [a, b]$ such that

$$a < a_1 < b_1 < b, \; b_1 - a_1 < 1 \text{ and } \operatorname{osc}(f;[a_1,b_1]) < 1.$$

In view of Lemma 1.2.6, $f \in \mathscr{R}[a_1, b_1]$.. Hence a further appeal to Lemma 1.2.7 shows that there is a closed interval $[a_2, b_2] \subset [a_1, b_1]$ such that

$$a_1 < a_2 < b_2 < b_1, \; b_2 - a_2 < 2^{-1} \text{ and } \operatorname{osc}(f;[a_2,b_2]) < 2^{-1}.$$

Continuing in this way, and allowing $a_0 := a$, $b_0 := b$, we see that there is a nested sequence $([a_n, b_n])$ of bounded, closed intervals such that, for each $n \in \mathbf{N}$,

(i) $a_{n-1} < a_n < b_n < b_{n-1}$,

(ii) $b_n - a_n < n^{-1}$, and

(iii) $\operatorname{osc}(f;[a_n,b_n]) < n^{-1}$.

Applying the Nested Intervals Principle (see the Appendix, Theorem A.4.15), we see that there exists $c \in \mathbf{R}$ such that $\{c\} = \cap_{n=1}^{\infty}[a_n, b_n]$. It remains to show that f is continuous at c. Let $\varepsilon > 0$. There exists $m \in \mathbf{N}$ such that $m\varepsilon > 1$ and, evidently, for all $x \in [a_m, b_m]$,

$$|f(x) - f(c)| \le \operatorname{osc}(f;[a_m,b_m]) < m^{-1} < \varepsilon.$$

With $\delta = \min\{c - a_m, b_m - c\}$, it follows that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$. Thus f is continuous at c. □

Definition 1.2.9 Let $\mathscr{I}$ be a non-empty interval in $\mathbf{R}$. A function $\phi : \mathscr{I} \to \mathbf{R}$ is said to satisfy a **Lipschitz condition** (**on** $\mathscr{I}$) if there exists $c > 0$ such that for all $s, t \in \mathscr{I}$,

$$|\phi(s) - \phi(t)| \le c\,|s - t|\,.$$

With this property, ϕ is also described as a **Lipschitz-continuous function** (**on** $\mathscr{I}$); the number c is said to be a **Lipschitz constant for** ϕ.

Theorem 1.2.10 *Suppose that $f \in \mathscr{R}[a, b]$ and that $f([a, b]) \subset [\alpha, \beta]$; let $\phi : [\alpha, \beta] \to \mathbf{R}$ be a Lipschitz-continuous function on $[\alpha, \beta]$, and let $h = \phi \circ f$. Then $h \in \mathscr{R}[a, b]$.*

Proof Let c be a Lipschitz constant for ϕ. By Exercise 1.2.14/6, ϕ is continuous (uniformly continuous) on $[\alpha, \beta]$. Hence h is bounded. Let $P = \{x_0, x_1, \ldots, x_n\} \in \mathscr{P}[a, b]$ and write $I_j = [x_{j-1}, x_j]$ $(j = 1, 2, \ldots, n)$. By Exercise 1.1.10/2,

$$\begin{aligned} U(P, h) - L(P, h) &= \sum_{j=1}^{n} \operatorname{osc}(h; I_j)\Delta x_j \leq c \sum_{j=1}^{n} \operatorname{osc}(f; I_j)\Delta x_j \\ &= c\,(U(P, f) - L(P, f))\,. \end{aligned}$$

Application of this inequality to the members of a sequence (P_n) of partitions of $[a, b]$ with $w(P_n) \to 0$ now shows, with the help of Theorem 1.1.7, that $h \in \mathscr{R}[a, b]$. ◻

Corollary 1.2.11 *If f is in $\mathscr{R}[a, b]$, so are $|f|$ and f^2.*

Proof Put $K = \sup\{|f(x)| : a \leq x \leq b\}$ and let $\mathscr{I} = [-K, K]$. Then, for all $s, t \in \mathscr{I}$,

$$||s| - |t|| \leq |s - t|\,, \quad \left|s^2 - t^2\right| \leq 2K\,|s - t|\,.$$

The maps $t \longmapsto |t|$ and $t \longmapsto t^2$ each satisfy a Lipschitz condition on $\mathscr{I}$ and so appeal to Theorem 1.2.10 gives the result. ◻

Theorem 1.2.10 enables us to generate new Riemann-integrable functions from functions known already to be Riemann-integrable. The next theorem goes further along this particular line and includes Theorem 1.2.10 as a special case. The condition that ϕ is a Lipschitz-continuous function is relaxed, simply requiring it to be continuous. The following lemma which, loosely speaking, asserts that every continuous real-valued function on a bounded, closed interval is 'almost' Lipschitz-continuous, paves the way for the relaxation.

Lemma 1.2.12 *Let $\phi : [\alpha, \beta] \to \mathbf{R}$ be continuous and let $\varepsilon > 0$. Then there exists $c > 0$ such that, for all $s, t \in [\alpha, \beta]$,*

$$|\phi(s) - \phi(t)| < \varepsilon + c\,|s - t|\,.$$

Proof To obtain a contradiction, we suppose the conclusion false. Then there exist $\varepsilon > 0$ and sequences (s_n), (t_n) in $[\alpha, \beta]$ such that for all $n \in \mathbf{N}$,

$$|\phi(s_n) - \phi(t_n)| \geq \varepsilon + n\,|s_n - t_n|\,.$$

By the Bolzano-Weierstrass theorem, there are points $s, t \in [\alpha, \beta]$ and subsequences $(s_{k(n)})$, $(t_{k(n)})$ of (s_n), (t_n) such that $\lim_{n\to\infty} s_{k(n)} = s$, $\lim_{n\to\infty} t_{k(n)} = t$. Evidently $\left|\phi(s_{k(n)}) - \phi(t_{k(n)})\right| \geq \varepsilon$; and since ϕ is continuous, we may let $n \to \infty$ and obtain $|\phi(s) - \phi(t)| \geq \varepsilon$, which implies that $s \neq t$. However, we then have

$$\begin{aligned} |\phi(s) - \phi(t)| &= \lim_{n\to\infty} \left|\phi(s_{k(n)}) - \phi(t_{k(n)})\right| \geq \lim_{n\to\infty} k(n)\left|s_{k(n)} - t_{k(n)}\right| \\ &= \lim_{n\to\infty} k(n) \lim_{n\to\infty} \left|s_{k(n)} - t_{k(n)}\right| = \infty, \end{aligned}$$

which is impossible. The result claimed thus holds. □

Theorem 1.2.13 *Suppose that* $f \in \mathscr{R}[a,b]$ *and that* $f([a,b]) \subset [\alpha,\beta]$; *let* $\phi : [\alpha,\beta] \to \mathbf{R}$ *be continuous and set* $h = \phi \circ f$. *Then* $h \in \mathscr{R}[a,b]$.

Proof Since ϕ is continuous, h is bounded. Let $\varepsilon > 0$. By Lemma 1.2.12, there exists $c > 0$ such that, for all $s, t \in [\alpha, \beta]$,

$$|\phi(s) - \phi(t)| < \varepsilon + c\,|s - t|\,.$$

Let (P_n) be a sequence of partitions of $[a,b]$ with $w(P_n) \to 0$. Then for all $n \in \mathbf{N}$,

$$U(P_n, h) - L(P_n, h) \le \varepsilon(b-a) + c\,\{U(P_n, f) - L(P_n, f)\}\,,$$

and by letting $n \to \infty$ we obtain, since $f \in \mathscr{R}[a,b]$,

$$\overline{\int_a^b} h - \underline{\int_a^b} h \le \varepsilon(b-a).$$

As this holds for all $\varepsilon > 0$, it follows that $\overline{\int_a^b} h = \underline{\int_a^b} h$, and the proof is complete. □

Note that Corollary 1.2.11 can be obtained from Theorem 1.2.13 even more directly than before.

Exercise 1.2.14

1. Let $f : [a,b] \to \mathbf{R}$ be continuous on $[a,b]$. Use the Bolzano-Weierstrass theorem to show directly that
 (i) f is bounded.
 (ii) f attains its bounds; that is, there exist $c, d \in [a,b]$ such that

$$f(c) = \inf_{x \in [a,b]} f(x), \; f(d) = \sup_{x \in [a,b]} f(x).$$

2. Let $f : (0,1] \to \mathbf{R}$ be defined by $f(x) = \cos(\pi/x)$ $(0 < x \le 1)$. Prove that f is continuous but not uniformly continuous on $(0,1]$.
3. Let $f : (0,1] \to \mathbf{R}$ be uniformly continuous on $(0,1]$. Through either a proof or exhibition of a counterexample, decide whether or not f is necessarily bounded.
4. Let $f : [0,1] \to \mathbf{R}$ be defined by $f(x) = x$ if $x = 1/n$ for some $n \in \mathbf{N}$, $f(x) = 0$ otherwise. Show that $f \in \mathscr{R}[0,1]$ and that $\int_0^1 f = 0$. [Hint: partition the interval $[0,1]$ into n^2 subintervals of equal length.]
5. Let $\mathscr{I}$ be a non-empty interval in $\mathbf{R}$ and let $f : \mathscr{I} \to \mathbf{R}$ be defined by

$$f(x) = x^2.$$

 Show that f satisfies a Lipschitz condition on $\mathscr{I}$ if, and only if, $\mathscr{I}$ is bounded.

6. Let $\mathscr{I}$ be a non-empty interval in $\mathbf{R}$ and let $f : \mathscr{I} \to \mathbf{R}$ be a Lipschitz-continuous function on $\mathscr{I}$. Show that f is uniformly continuous on $\mathscr{I}$.
7. Let $f : [a, b] \to \mathbf{R}$ be differentiable. Show that f is Lipschitz-continuous on $[a, b]$ if, and only if, its derivative, f', is bounded on $[a, b]$.
8. (a) Give an example of a Lipschitz-continuous function on [0, 1] which is not differentiable on [0, 1].
   (b) Let $f : [0, 1] \to \mathbf{R}$ be defined by $f(x) = x^2 \sin(x^{-2})$ if $0 < x \leq 1$; $f(0) = 0$. Show that f does not satisfy a Lipschitz condition on [0, 1].
9. Give an example of a function $f \in \mathscr{B}[0, 1] \backslash \mathscr{R}[0, 1]$ which has a point of continuity in the open interval (0, 1).
10. Let $f \in \mathscr{R}[a, b]$ and let $x \in [a, b]$. Prove that there is a sequence (x_n) of distinct points in $[a, b]$ such that
    (i) $\lim_{n\to\infty} x_n = x$, and
    (ii) each x_n is a point of continuity of f.

1.3 Properties of the Integral

In this section we establish numerous useful properties of the Riemann integral. We begin with upper and lower integrals.

Theorem 1.3.1 *Let $f, g \in \mathscr{B}[a, b]$ and let $\lambda \in \mathbf{R}$. Then:*

(i) $\overline{\int_a^b} f + \overline{\int_a^b} g \geq \overline{\int_a^b} (f + g) \geq \underline{\int_a^b} (f + g) \geq \underline{\int_a^b} f + \underline{\int_a^b} g$;

(ii) *if* $\lambda \geq 0$, *then* $\overline{\int_a^b} \lambda f = \lambda \overline{\int_a^b} f$ *and* $\underline{\int_a^b} \lambda f = \lambda \underline{\int_a^b} f$;

(iii) *if* $\lambda < 0$, *then* $\overline{\int_a^b} \lambda f = \lambda \underline{\int_a^b} f$ *and* $\underline{\int_a^b} \lambda f = \lambda \overline{\int_a^b} f$;

(iv) *if* $f(t) \geq g(t)$ *for all* $t \in [a, b]$, *then*

$$\overline{\int_a^b} f \geq \overline{\int_a^b} g \text{ and } \underline{\int_a^b} f \geq \underline{\int_a^b} g;$$

(v) *if* $f(t) = g(t)$ *at all but a finite number of points of* $[a, b]$, *then*

$$\overline{\int_a^b} f = \overline{\int_a^b} g \text{ and } \underline{\int_a^b} f = \underline{\int_a^b} g \, ;$$

(vi) *if* $f(t) \geq 0$ *for all* $t \in [a, b]$ *and* $f(c) > 0$ *at some point* $c \in [a, b]$ *at which* f *is continuous, then* $\underline{\int_a^b} f > 0$.