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Geometric Analysis of the Bergman Kernel and Metric

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Dedicated to the memory of Stefan Bergman,
an extraordinarily profound and original
mathematician

Preface

The Bergman kernel and metric have been a seminal part of geometric analysis and partial differential equations since their invention by Stefan Bergman in 1922. Applications to holomorphic mappings, to function theory, to partial differential equations, and to differential geometry have kept the techniques plugged into the mainstream of mathematics for 90 years.

The Bergman kernel is based on a very simple idea: that the square-integrable holomorphic functions on a bounded domain in that complex space form a Hilbert space. Moreover, a simple formal argument shows that Hilbert space possesses a so-called reproducing kernel. This is an integration kernel which reproduces each element of the space. The kernel has wonderful invariance properties, leading to the Bergman metric. The Bergman kernel and metric have developed into powerful tools for function theory, analysis, differential geometry, and partial differential equations. The purpose of this book is to exposit this theory (particularly in the context of several complex variables), examine its key features, and bring the reader up to speed with some of the latest developments.

Bergman wrote several books about his kernel and contributed mightily to its development. The idea caught on widely, and the kernel became a standard device in the field. The Bergman metric was the first-ever Kähler metric, and that in turn spawned the vital subject of complex differential geometry. Ahlfors (in one variable), Chern (in several variables), Greene–Wu, and many others played a decisive role in this development.

Bergman's ideas received a major boost in the 1970s when Charles Fefferman did his Fields Medal-winning work on the boundary behavior of biholomorphic mappings. The key device in his analysis was the Bergman kernel and metric. Since then, a myriad of workers, from Bell and Ligocka to Webster to Greene–Krantz, Krantz–Li, Kim–Krantz, Isaev–Krantz, and many others, have developed and extended Bergman's theory. It is now part of the lingua franca of complex analysis, and the technique of reproducing kernels which it has spawned is part of every analyst's toolkit.

Fefferman's work inspired many others to examine the utility of the Bergman theory in the study of biholomorphic mappings. Bell's condition R, formulated in terms of regularity properties of the Bergman projection, has proved to be an influential and powerful weapon in the subject. In turn, condition R can be formulated in terms of the regularity theory of subelliptic partial differential equations, and this connection has had a key influence on the directions of research.

The Bergman metric was the first "universal" (in the sense that it can be constructed on virtually any domain) example of a metric that is invariant under biholomorphic mappings. [The Poincaré metric was of course the primordial example on the disc.] Today there are the Kobayashi–Royden metric, the Sibony metric, the Carathéodory metric, and many others. This is a useful tool in geometric analysis and function theory.

The connections of the Bergman kernel with partial differential equations, especially the extremal properties of the kernel and metric, are profound. Bergman himself explored applications of his theory to elliptic partial differential equations. Today we see the Bergman kernel as inextricably linked with the $\overline{\partial}$-Neumann problem. This link played a vital role in Fefferman's work and later proved crucial to Greene–Krantz and many of the other workers in the subject. Certainly Donald Spencer and J. J. Kohn were the pioneers of this symbiosis. Today the interaction is prospering. Another development is that the Bergman theory enjoys connections with the Monge–Ampère equation. That nonlinear partial differential equation contains important information about biholomorphic mappings and about the construction of geometries.

Connections with harmonic analysis are another exciting, and relatively new, aspect of the Bergman paradigm. Coifman–Rochberg–Weiss used the Bergman kernel in their proof of the H^1/BMO duality theorem on the ball, and Krantz–Li exploited it further in their study on strictly pseudoconvex domains. Many of the natural artifacts of harmonic analysis—including approach regions for Fatou theorems—are most propitiously formulated in terms of Bergman geometry or the boundary asymptotics of the kernel. The Bergman kernel is now a standard artifact of the harmonic analysis of several complex variables.

This text will in fact be a thoroughgoing treatment of all the basic analytic and geometric aspects of Bergman's theory. This will include

- Definitions of the Bergman kernel
- Definition and basic properties of the Bergman metric
- Calculation of the Bergman kernel and metric
- Invariance properties of the kernel and metric
- Boundary asymptotics of the kernel and metric
- Asymptotic expansions for the Bergman kernel
- Applications to function theory
- Applications to geometry
- Applications to partial differential equations
- Interpretations in terms of functional analysis
- The geometry of the Bergman metric
- Curvature of the Bergman metric
- The Bergman kernel and metric on manifolds

There are a few recent treatises on the Bergman kernel, notably those by Hedenmalm–Korenblum–Zhu and Duren–Schuster. But these books concentrate on the one-variable theory; they are also oriented towards the functional analysis aspects of the Bergman kernel. Our focus instead is the geometry of several complex variables and the contexts of real analysis, complex analysis, harmonic analysis, and differential geometry. This puts Bergman's ideas into a much broader arena and provides many more opportunities for applications and illustrations. We will certainly touch on the functional analysis properties of the Bergman projection, but these will not be our main focus. We shall also cover selected topics of the one complex variable theory. There is little overlap between this book and the two books cited above.

We would also be remiss not to mention the book of Ma and Marinescu on the Kähler geometry aspects of the Bergman theory. Certainly the Bergman metric was the very first Kähler metric, and this in turn has spawned the active and fruitful area of multivariable complex differential geometry. Various parts of the present book touch on this Kähler theory.

Lurking in the background behind Fefferman's biholomorphic mapping theorem were Bergman representative coordinates—yet another outgrowth of the Bergman kernel and metric. This is a much-underappreciated aspect of the theory and one that we shall treat in detail in the text. In fact there are many aspects of the Bergman theory that tend only to be known to experts and are not readily accessible in the literature. We intend to treat many of those. Several of the topics in this text appear here for the first time in book form.

We intend this to be a book for students as well as seasoned researchers. All needed background will be provided. The reader is only assumed to have had a solid course in complex variables and some basic background in real and functional analysis. A little exposure to geometry will be helpful, but is not a requirement. There are many illustrative examples and some useful figures. The book abounds with useful and instructive calculations, many of which cannot be found elsewhere. Every chapter ends with a selection of exercises, which should serve to help the reader get more directly involved in the subject matter. It will cause him/her to consult the literature, to calculate, and to learn by doing.

The book will help the novice reader to see how analysis is used in practice and how it can be evolved into a seminal tool for research. It is important for the student to see fundamental mathematics used in vitro in order to understand how research develops and grows.

It is a pleasure to thank E. M. Stein for introducing me to the Bergman kernel and Robert E. Greene for teaching me the geometric aspects. I have had many collaborators in my study of the kernel, and I offer them all my gratitude. I thank my editor, Elizabeth Loew, for her constant enthusiasm and support. And I thank the several referees for this book, who contributed a wealth of ideas and information.

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Contents

Chapter 1
Introductory Ideas

In the early days of functional analysis—the early twentieth century—people did not yet know what a Banach space was nor a Hilbert space. They frequently studied a *particular* complete, infinite-dimensional space from a more abstract point of view. The most common space to be studied in this regard was of course L^2. It was when Stefan Bergman took a course from Erhard Schmidt on L^2 of the unit interval I that he conceived of the idea of the Bergman space of square-integrable holomorphic functions on the unit disc D. And the rest is history.

It is important for the Bergman theory that his space of holomorphic functions has an inner product structure and that it is complete. The first of these properties follows from the fact that it is a subspace of L^2; the second follows from a fundamental inequality that we shall consider in the next section.

1.1 The Bergman Kernel

It is difficult to create an explicit integral formula, with holomorphic reproducing kernel, for holomorphic functions on an arbitrary domain in $\mathbb{C}^n$.[1] Classical studies which perform such constructions tend to concentrate on domains having a great deal of symmetry (see, for instance, [HUA]). We now examine one of several non-constructive approaches to this problem. This circle of ideas, due to Bergman [BER1] and to Szegő [SZE] (some of the ideas presented here were anticipated by the thesis of Bochner [BOC1]), will later be seen to have profound applications to the boundary regularity of holomorphic mappings.

Bungart [BUN] and Gleason [GLE] have shown that *any* bounded domain in $\mathbb{C}^n$ will have a reproducing kernel for holomorphic functions such that the kernel itself is holomorphic in the free variable. In other words, the formula has the form

[1]Here a *domain* is a connected, open set.

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$$f(z) = \int_\Omega f(\zeta) K(z, \zeta)\, \mathrm{d}V(\zeta),$$

and K is holomorphic in the z variable. Of course Bungart's and Gleason's proofs are highly nonconstructive, and one can say almost nothing about the actual form of the kernel K. The venerable Bochner–Martinelli kernel is easily constructed on any bounded domain with reasonable boundary (just as an application of Stokes's theorem) and the kernel is *explicit*—just like the Cauchy kernel in one complex variable. Also the kernel is the same for every domain. But the Bochner–Martinelli kernel is definitely *not* holomorphic in the free variable. On the other hand, Henkin [HEN], Kerzman [KER1], E. Ramirez [RAMI], and Grauert–Lieb [GRL] have given very explicit constructions of reproducing kernels on strictly pseudoconvex domains (see the definition below). And their kernels *are* holomorphic in the z variable. This matter is treated in [KRA1, Chap. 10].

In fact this last described result was considered to be quite a dramatic advance. For Henkin, Kerzman, Ramirez, and Grauert–Lieb provided us with a fairly explicit kernel, with an explicit and measurable singularity, that can not only reproduce but also create holomorphic functions. Such a kernel is very much like the Cauchy kernel in one complex variable. Thus at least on strictly pseudoconvex domains, we can perform many of the activities to which we are accustomed from the function theory of one complex variable. We can get formulas for derivatives of holomorphic functions, we can analyze power series, we can consider an analogue of the Cauchy transform, and (perhaps most importantly) we can write down solution operators for the $\overline{\partial}$ problem. People were optimistic that these new integral formulas would give a shot in the arm to the theory of function algebras—that they would now be able to study $H^\infty(\Omega)$ and $A(\Omega)$ on a variety of domains in $\mathbb{C}^n$ (see [GAM, Chap. II, IV] for the role model in $\mathbb{C}^1$). But this turned out to be too difficult.

The Bergman kernel is a canonical kernel that can be defined on any bounded domain. It has wonderful invariance properties and is a powerful tool for geometry and analysis. But it is difficult to calculate explicitly.

In this section we will see some of the invariance properties of the Bergman kernel. This will lead in later sections to the definition of the Bergman metric (in which all biholomorphic mappings become isometries) and to such other canonical constructions as representative coordinates. The Bergman kernel has certain extremal properties that make it a powerful tool in the theory of partial differential equations (see Bergman and Schiffer [BES]). Also the form of the singularity of the Bergman kernel (calculable for some interesting classes of domains) explains many phenomena of the function theory of several complex variables.

Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain (it is possible, but often tricky, to treat unbounded domains as well). Here a *domain* is a connected, open set. If the domain is smoothly bounded, then we may think of it as specified by a *defining function*:

$$\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}.$$

It is customary to require that $\nabla\rho \neq 0$ on $\partial\Omega$. One can demonstrate the existence of a definiting function by using the implicit function theorem. See [KRPA2] for the latter and [KRA1] for a detailed consideration of defining functions.

Given a domain Ω as described in the last paragraph and a point $P \in \partial\Omega$, we say that w is a *complex tangent vector* at P and write $w \in \mathcal{T}_P(\partial\Omega)$ if

$$\sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j}(P) w_j = 0 .$$

The point P is said to be *strongly pseudoconvex* if

$$\sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}(P) w_j \overline{w}_k > 0$$

for $0 \neq w \in \mathcal{T}_P(\partial\Omega)$. In fact a little elementary analysis shows that we can write the defining property of strong pseudoconvexity as

$$\sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}(P) w_j \overline{w}_k \geq C|w|^2$$

and make the estimate uniform when P ranges over a compact, strongly pseudoconvex boundary neighborhood of Ω. Again, the book [KRA1, Chap. 3] has extensive discussion of the notion of strong pseudoconvexity.

Now let us return to the Bergman theory. Let $\mathrm{d}V$ denote the Lebesgue volume measure on Ω. Define the *Bergman space*

$$A^2(\Omega) = \left\{ f \text{ holomorphic on } \Omega : \int_\Omega |f(z)|^2 \,\mathrm{d}V(z)^{1/2} \equiv \|f\|_{A^2(\Omega)} < \infty \right\} .$$

Of course we equip the Bergman space with the inner product

$$\langle f, g \rangle = \int_\Omega f(z)\overline{g}(z) \,\mathrm{d}V(z).$$

Lemma 1.1.1. *Let $K \subseteq \Omega \subseteq \mathbb{C}^n$ be compact. There is a constant $C_K > 0$, depending on K and on n, such that*

$$\sup_{z \in K} |f(z)| \leq C_K \|f\|_{A^2(\Omega)}, \quad \text{all } f \in A^2(\Omega).$$

Proof. Since K is compact, there is an $r(K) = r > 0$ so that, for any $z \in K$, $B(z, r) \subseteq \Omega$. Here $B(z, r)$ is the usual Euclidean ball with center z and radius r.

Therefore for each $z \in K$ and $f \in A^2(\Omega)$, the mean-value property for holomorphic functions implies that

$$\begin{aligned}
|f(z)| &= \left| \frac{1}{V(B(z,r))} \int_{B(z,r)} f(t)\,\mathrm{d}V(t) \right| \\
&= \left| \frac{1}{V(B(z,r))} \int f(t)\chi_{B(z,r)}(t)\,\mathrm{d}V(t) \right| \\
&\leq (V(B(z,r)))^{-1/2} \|f\|_{L^2(B(z,r))} \\
&\leq C(n) r^{-n} \|f\|_{A^2(\Omega)} \\
&\equiv C_K \|f\|_{A^2(\Omega)}. \qquad \square
\end{aligned}$$

Lemma 1.1.2. *The space $A^2(\Omega)$ is a Hilbert space with the inner product $\langle f, g \rangle \equiv \int_\Omega f(z)\overline{g(z)}\,\mathrm{d}V(z)$.*

Proof. Everything is clear except for completeness. Let $\{f_j\} \subseteq A^2$ be a sequence that is Cauchy in norm. Since L^2 is complete there is an L^2 limit function f. We need to see that f is holomorphic. But Lemma 1.1.1 yields that norm convergence implies normal convergence (i.e., uniform convergence on compact sets). Certainly holomorphic functions are closed under normal limits (just use the Cauchy theory of one complex variable). Therefore f is holomorphic and $A^2(\Omega)$ is complete. $\square$

Lemma 1.1.3. *For each fixed $z \in \Omega$, the functional*

$$\Phi_z : f \mapsto f(z), \quad f \in A^2(\Omega)$$

is a continuous linear functional on $A^2(\Omega)$.

Proof. This is immediate from Lemma 1.1.1 if we take K to be the singleton $\{z\}$. $\square$

We may now apply the Riesz representation theorem to see that there is an element $K_z \in A^2(\Omega)$ such that the linear functional Φ_z is represented by inner product with K_z : if $f \in A^2(\Omega)$, then, for all $z \in \Omega$, we have

$$f(z) = \Phi_z(f) = \langle f, K_z \rangle.$$

Definition 1.1.4. The *Bergman kernel* is the function $K(z,\zeta) = K_\Omega(z,\zeta) \equiv \overline{K_z(\zeta)}$, $z, \zeta \in \Omega$. It has the reproducing property

$$f(z) = \int_\Omega K(z,\zeta) f(\zeta)\,\mathrm{d}V(\zeta), \quad \forall f \in A^2(\Omega).$$

Proposition 1.1.5. *The Bergman kernel $K(z,\zeta)$ is conjugate symmetric: $K(z,\zeta) = \overline{K(\zeta,z)}$.*

Proof. By its very definition, $\overline{K(\zeta,\cdot)} \in A^2(\Omega)$ for each fixed ζ. Therefore the reproducing property of the Bergman kernel gives

$$\int_\Omega K(z,t)\overline{K(\zeta,t)}\,\mathrm{d}V(t) = \overline{K(\zeta,z)}.$$

On the other hand,

$$\begin{aligned}\int_\Omega K(z,t)\overline{K(\zeta,t)}\,\mathrm{d}V(t) &= \overline{\int K(\zeta,t)\overline{K(z,t)}\,\mathrm{d}V(t)} \\ &= \overline{\overline{K(z,\zeta)}} = K(z,\zeta). \qquad \square\end{aligned}$$

Proposition 1.1.6. *The Bergman kernel is uniquely determined by the properties that it is an element of $A^2(\Omega)$ in z, is conjugate symmetric, and reproduces $A^2(\Omega)$.*

Proof. Let $K'(z,\zeta)$ be another such kernel. Then

$$\begin{aligned}K(z,\zeta) = \overline{K(\zeta,z)} &= \int K'(z,t)\overline{K(\zeta,t)}\,\mathrm{d}V(t) \\ &= \overline{\int K(\zeta,t)\overline{K'(z,t)}\,\mathrm{d}V(t)} \\ &= \overline{\overline{K'(z,\zeta)}} = K'(z,\zeta). \qquad \square\end{aligned}$$

Since $L^2(\Omega)$ is a separable Hilbert space then so is its subspace $A^2(\Omega)$. Thus there is a countable, complete orthonormal basis $\{\phi_j\}_{j=1}^\infty$ for $A^2(\Omega)$.

Proposition 1.1.7. *Let L be a compact subset of Ω. Then the series*

$$\sum_{j=1}^\infty \phi_j(z)\overline{\phi_j(\zeta)}$$

sums uniformly on $L \times L$ to the Bergman kernel $K(z,\zeta)$.

Proof. By the Riesz–Fischer and Riesz representation theorems, we obtain

$$\begin{aligned}\sup_{z\in L}\left(\sum_{j=1}^\infty |\phi_j(z)|^2\right)^{1/2} &= \sup_{z\in L}\left\|\{\phi_j(z)\}_{j=1}^\infty\right\|_{\ell^2} \\ &= \sup_{\substack{\|\{a_j\}\|_{\ell^2}=1 \\ z\in L}}\left|\sum_{j=1}^\infty a_j\phi_j(z)\right| \\ &= \sup_{\substack{\|f\|_{A^2}=1 \\ z\in L}} |f(z)| \\ &\le C_L\,. \qquad (1.1.7.1)\end{aligned}$$

In the last inequality we have used Lemma 1.1.1. Therefore

$$\sum_{j=1}^{\infty} \left| \phi_j(z)\overline{\phi_j(\zeta)} \right| \le \left(\sum_{j=1}^{\infty} |\phi_j(z)|^2 \right)^{1/2} \left(\sum_{j=1}^{\infty} |\phi_j(\zeta)|^2 \right)^{1/2}$$

and the convergence is uniform over $z, \zeta \in L$. For fixed $z \in \Omega$, (1.1.7.1) shows that $\{\phi_j(z)\}_{j=1}^{\infty} \in \ell^2$. Hence we have that $\sum \phi_j(z)\overline{\phi_j(\zeta)} \in \overline{A^2(\Omega)}$ as a function of ζ. Let the sum of the series be denoted by $K'(z,\zeta)$. Notice that K' is conjugate symmetric by its very definition. Also, for $f \in A^2(\Omega)$, we have

$$\int K'(\cdot,\zeta) f(\zeta)\, \mathrm{d}V(\zeta) = \sum \hat{f}(j)\phi_j(\cdot) = f(\cdot),$$

where convergence is in the Hilbert space topology. (Here $\hat{f}(j)$ is the jth Fourier coefficient of f with respect to the basis $\{\phi_j\}$.) But Hilbert space convergence dominates pointwise convergence (Lemma 1.1.1) so

$$f(z) = \int K'(z,\zeta) f(\zeta)\, \mathrm{d}V(\zeta), \quad \text{all } f \in A^2(\Omega).$$

Therefore K' is the Bergman kernel. □

Remark 1.1.8. It is worth noting explicitly that the proof of Proposition 1.1.7 shows that

$$\sum \phi_j(z)\overline{\phi_j(\zeta)}$$

equals the Bergman kernel $K(z,\zeta)$ *no matter what the choice* of complete orthonormal basis $\{\phi_j\}$ for $A^2(\Omega)$. This can be very useful information in practice. □

Proposition 1.1.9. *If Ω is a bounded domain in $\mathbb{C}^n$, then the mapping*

$$P : f \mapsto \int_{\Omega} K(\cdot,\zeta) f(\zeta)\, \mathrm{d}V(\zeta)$$

is the Hilbert space orthogonal projection of $L^2(\Omega,\, \mathrm{d}V)$ onto $A^2(\Omega)$. We call P the Bergman projection.

Proof. Notice that P is idempotent and self-adjoint and that $A^2(\Omega)$ is precisely the set of elements of L^2 that are fixed by P. □

Definition 1.1.10. Let $\Omega \subseteq \mathbb{C}^n$ be a domain and let $f : \Omega \to \mathbb{C}^n$ be a *holomorphic mapping*, that is, $f(z) = (f_1(z), \ldots, f_n(z))$ with $f_1, \ldots, f_n$ holomorphic on Ω. Let $w_j = f_j(z)$, $j = 1, \ldots, n$. Then the *holomorphic Jacobian matrix* of f is the matrix

$$J_{\mathbb{C}} f = \frac{\partial(w_1, \ldots, w_n)}{\partial(z_1, \ldots, z_n)} .$$

Write $z_j = x_j + iy_j, w_k = \xi_k + i\eta_k$, $j, k = 1, \ldots, n$. Then the *real Jacobian matrix* of f is the matrix

$$J_{\mathbb{R}} f = \frac{\partial(\xi_1, \eta_1, \ldots, \xi_n, \eta_n)}{\partial(x_1, y_1, \ldots, x_n, y_n)} .$$

Proposition 1.1.11. *With notation as in the definition, we have*

$$\det J_{\mathbb{R}} f = |\det J_{\mathbb{C}} f|^2$$

whenever f is a holomorphic mapping.

Proof. We exploit the functoriality of the Jacobian. Let $w = (w_1, \ldots, w_n) = f(z) = (f_1(z), \ldots, f_n(z))$. Write $z_j = x_j + iy_j, w_j = \xi_j + i\eta_j$, $j = 1, \ldots, n$. Then, using the fact that f is holomorphic,

$$d\xi_1 \wedge d\eta_1 \wedge \cdots \wedge d\xi_n \wedge d\eta_n = (\det J_{\mathbb{R}} f(x, y)) dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n. \quad (1.1.11.1)$$

On the other hand,

$$\begin{aligned}
& d\xi_1 \wedge d\eta_1 \wedge \cdots \wedge d\xi_n \wedge d\eta_n \\
&= \frac{1}{(2i)^n} d\overline{w}_1 \wedge dw_1 \wedge \cdots \wedge d\overline{w}_n \wedge dw_n \\
&= \frac{1}{(2i)^n} \overline{(\det J_{\mathbb{C}} f(z))} (\det J_{\mathbb{C}} f(z)) d\overline{z}_1 \wedge dz_1 \wedge \cdots \wedge d\overline{z}_n \wedge dz_n \\
&= |\det J_{\mathbb{C}} f(z)|^2 dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n. \qquad (1.1.11.2)
\end{aligned}$$

Equating (1.1.11.1) and (1.1.11.2) gives the result. □

Exercise for the Reader: Prove Proposition 1.1.11 using only matrix theory (no differential forms). This will give rise to a great appreciation for the theory of differential forms (see Bers [BERS, Chap. 7] for help).

Now we can prove the holomorphic implicit function theorem:

Theorem 1.1.12. *Let $f_j(w,z)$, $j = 1,\ldots,m$ be holomorphic functions of $(w,z) = ((w_1,\ldots,w_m),(z_1,\ldots,z_n))$ near a point $(w^0,z^0) \in \mathbb{C}^m \times \mathbb{C}^n$. Assume that*

$$f_j(w^0,z^0) = 0, \quad j = 1,\ldots,m,$$

and that

$$\det \left(\frac{\partial f_j}{\partial w_k}\right)_{j,k=1}^m \neq 0 \text{ at } (w^0,z^0).$$

Then the system of equations

$$f_j(w,z) = 0 \ , \ \ j = 1,\ldots,m,$$

has a unique holomorphic solution $w(z)$ in a neighborhood of z^0 that satisfies $w(z^0) = w^0$.

Proof. We rewrite the system of equations as

$$\operatorname{Re} f_j(w,z) = 0 \ , \ \ \operatorname{Im} f_j(w,z) = 0$$

for the $2m$ real variables $\operatorname{Re} w_k, \operatorname{Im} w_k, k = 1,\ldots,m$. By Proposition 1.1.11, the determinant of the Jacobian over $\mathbb{R}$ of this new system is the modulus squared of the determinant of the Jacobian over $\mathbb{C}$ of the old system. By our hypothesis, this number is nonvanishing at the point (w^0,z^0). Therefore the classical implicit function theorem (see Rudin [RUD1] or [KRPA2]) implies that there exist C^1 functions $w_k(z), k = 1,\ldots,m$, with $w(z^0) = w^0$ and that solve the system. Our job is to show that these functions are in fact holomorphic. When properly viewed, this is purely a problem of geometric algebra:

Applying exterior differentiation to the equations

$$0 = f_j(w(z),z) \ , \ \ j = 1,\ldots,m,$$

yields that

$$0 = \mathrm{d}f_j = \sum_{k=1}^m \frac{\partial f_j}{\partial w_k}\mathrm{d}w_k + \sum_{k=1}^m \frac{\partial f_j}{\partial z_k}\mathrm{d}z_k.$$

There are no $\mathrm{d}\bar{z}_j$'s and no $\mathrm{d}\overline{w}_k$'s because the f_j's are holomorphic.

The result now follows from linear algebra only: The hypothesis on the determinant of the matrix $(\partial f_j/\partial w_k)$ implies that we can solve for $\mathrm{d}w_k$ in terms of $\mathrm{d}z_j$. Therefore w is a holomorphic function of z. □

A holomorphic mapping $f : \Omega_1 \to \Omega_2$ of domains $\Omega_1 \subseteq \mathbb{C}^n, \Omega_2 \subseteq \mathbb{C}^n$ is said to be *biholomorphic* if it is one-to-one, onto, and $\det J_{\mathbb{C}} f(z) \neq 0$ for every $z \in \Omega_1$.

Exercise for the Reader: Use Theorem 1.1.12 to prove that a biholomorphic mapping has a holomorphic inverse (hence the name).

Remark 1.1.13. It is true, but not at all obvious, that the nonvanishing of the Jacobian determinant is a superfluous condition in the definition of "biholomorphic mapping"; that is, the nonvanishing of the Jacobian follows from the univalence of the mapping. A proof of this assertion is sketched in Exercise 37 at the end of [KRA1, Chap. 11]. □

In what follows we shall frequently denote the Bergman kernel for a given domain Ω by K_Ω.

Proposition 1.1.14. *Let Ω_1, Ω_2 be domains in $\mathbb{C}^n$. Let $f : \Omega_1 \to \Omega_2$ be biholomorphic. Then*

$$\det J_{\mathbb{C}} f(z) K_{\Omega_2}(f(z), f(\zeta)) \det \overline{J_{\mathbb{C}} f(\zeta)} = K_{\Omega_1}(z, \zeta).$$

Proof. Let $\phi \in A^2(\Omega_1)$. Then, by change of variable,

$$\begin{aligned} &\int_{\Omega_1} \det J_{\mathbb{C}} f(z) K_{\Omega_2}(f(z), f(\zeta)) \det \overline{J_{\mathbb{C}} f(\zeta)} \phi(\zeta)\, \mathrm{d}V(\zeta) \\ &= \int_{\Omega_2} \det J_{\mathbb{C}} f(z) K_{\Omega_2}(f(z), \tilde{\zeta}) \det \overline{J_{\mathbb{C}} f(f^{-1}(\tilde{\zeta}))} \phi(f^{-1}(\tilde{\zeta})) \\ &\times \det J_{\mathbb{R}} f^{-1}(\tilde{\zeta})\, \mathrm{d}V(\tilde{\zeta}). \end{aligned}$$

By Proposition 1.1.11 this simplifies to

$$\det J_{\mathbb{C}} f(z) \int_{\Omega_2} K_{\Omega_2}(f(z), \tilde{\zeta}) \left\{ \left(\det J_{\mathbb{C}} f(f^{-1}(\tilde{\zeta})) \right)^{-1} \phi \left(f^{-1}(\tilde{\zeta}) \right) \right\} \mathrm{d}V(\tilde{\zeta}).$$

By change of variables, the expression in braces $\{\ \}$ is an element of $A^2(\Omega_2)$. So the reproducing property of K_{Ω_2} applies and the last line equals

$$= \det J_{\mathbb{C}} f(z) \left(\det J_{\mathbb{C}} f(z)\right)^{-1} \phi \left(f^{-1}(f(z)) \right) = \phi(z).$$

By the uniqueness of the Bergman kernel, the proposition follows. □

Proposition 1.1.15. *For $z \in \Omega \subset\subset \mathbb{C}^n$ it holds that $K_\Omega(z,z) > 0$.*

Proof. Now

$$K_\Omega(z,z) = \sum_{j=1}^{\infty} |\phi_j(z)|^2 \geq 0.$$

If in fact $K(z,z) = 0$ for some z, then $\phi_j(z) = 0$ for all j; hence, $f(z) = 0$ for every $f \in A^2(\Omega)$. This is absurd. □

Definition 1.1.16. For any bounded domain $\Omega \subseteq \mathbb{C}^n$, we define a Hermitian metric on Ω by

$$g_{ij}(z) = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K(z,z), \quad z \in \Omega.$$

This means that the square of the length of a tangent vector $\xi = (\xi_1, \ldots, \xi_n)$ at a point $z \in \Omega$ is given by

$$|\xi|^2_{B,z} = \sum_{i,j} g_{ij}(z)\xi_i \overline{\xi}_j.$$

The metric that we have defined is called the *Bergman metric*.

In a Hermitian metric $\{g_{ij}\}$, the length of a C^1 curve $\gamma : [0,1] \to \Omega$ is given by

$$\ell(\gamma) = \int_0^1 \left(\sum_{i,j} g_{i,j}(\gamma(t))\gamma_i'(t)\overline{\gamma_j'(t)} \right)^{1/2} \mathrm{d}t.$$

If P, Q are points of Ω, then their distance $d_\Omega(P,Q)$ in the metric is defined to be the infimum of the lengths of all piecewise C^1 curves connecting the two points.

Remark 1.1.17. It is not a priori obvious that the Bergman metric for a bounded domain Ω is given by a positive definite matrix at each point. We now outline a proof of this fact.

First we generate an orthonormal basis for the Bergman space. Fix $z_0 \in \Omega$. Let ϕ_0 be the (unique!) element of A^2 with $\phi_0(z_0)$ real, $\|\phi_0\| = 1$, and $\phi_0(z_0)$ maximal. (Why does such a ϕ_0 exist?) Let ϕ_1 be the (unique) element of A^2 with $\phi_1(z_0) = 0$, $(\partial\phi_1/\partial z_1)(z_0)$ real, $\|\phi_1\| = 1$, and $(\partial\phi_1/\partial z_1)(z_0)$ maximal. (Why does such a ϕ_1 exist?) Now ϕ_1 is orthogonal to ϕ_0, else ϕ_1 has nonzero projection on ϕ_0, leading to a contradiction. Continue this process to create an orthogonal system on Ω. Use Taylor series to see that it is complete. This circle of ideas comes from the elegant paper Kobayashi [KOB1].

Now let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain and let (g_{ij}) be its Bergman metric. Use the ideas in the last paragraph to prove that the matrix $(g_{ij}(z))$ is positive definite, each $z \in \Omega$. [*Hint:* The crucial fact is that, for each $z \in \Omega$ and each j, there is an element $f \in A^2(\Omega)$ such that $\partial f/\partial z_j(z) \neq 0$.] □