How to Solve It

A New Aspect of

Mathematical Method

G. POLYA

With a new foreword by John H. Conway

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From the Preface to the First Printing

A great discovery solves a great problem but there is a grain of discovery in the solution of an problem. Your problem may be modest; but if it challenges your curiosity and brings into play you inventive faculties, and if you solve it by your own means, you may experience the tension and enjo the triumph of discovery. Such experiences at a susceptible age may create a taste for mental work ar leave their imprint on mind and character for a lifetime.

Thus, a teacher of mathematics has a great opportunity. If he fills his allotted time with drilling h students in routine operations he kills their interest, hampers their intellectual development, ar misuses his opportunity. But if he challenges the curiosity of his students by setting them problen proportionate to their knowledge, and helps them to solve their problems with stimulating question he may give them a taste for, and some means of, independent thinking.

Also a student whose college curriculum includes some mathematics has a singular opportunit This opportunity is lost, of course, if he regards mathematics as a subject in which he has to earn s and so much credit and which he should forget after the final examination as quickly as possible. T opportunity may be lost even if the student has some natural talent for mathematics because he, everybody else, must discover his talents and tastes; he cannot know that he likes raspberry pie if l has never tasted raspberry pie. He may manage to find out, however, that a mathematics problem ma be as much fun as a crossword puzzle, or that vigorous mental work may be an exercise as desirable a fast game of tennis. Having tasted the pleasure in mathematics he will not forget it easily and the there is a good chance that mathematics will become something for him: a hobby, or a tool of h profession, or his profession, or a great ambition.

The author remembers the time when he was a student himself, a somewhat ambitious studer eager to understand a little mathematics and physics. He listened to lectures, read books, tried to tal in the solutions and facts presented, but there was a question that disturbed him again and again: "Ye the solution seems to work, it appears to be correct; but how is it possible to invent such a solutior Yes, this experiment seems to work, this appears to be a fact; but how can people discover such facts And how could I invent or discover such things by myself?" Today the author is teaching mathemati in a university; he thinks or hopes that some of his more eager students ask similar questions and l tries to satisfy their curiosity. Trying to understand not only the solution of this or that problem b also the motives and procedures of the solution, and trying to explain these motives and procedures others, he was finally led to write the present book. He hopes that it will be useful to teachers wl wish to develop their students' ability to solve problems, and to students who are keen on developir their own abilities.

Although the present book pays special attention to the requirements of students and teachers mathematics, it should interest anybody concerned with the ways and means of invention ar discovery. Such interest may be more widespread than one would assume without reflection. Tl space devoted by popular newspapers and magazines to crossword puzzles and other riddles seems show that people spend some time in solving unpractical problems. Behind the desire to solve this that problem that confers no material advantage, there may be a deeper curiosity, a desire understand the ways and means, the motives and procedures, of solution.

The following pages are written somewhat concisely, but as simply as possible, and are based on long and serious study of methods of solution. This sort of study, called *heuristic* by some writers, not in fashion nowadays but has a long past and, perhaps, some future.

Studying the methods of solving problems, we perceive another face of mathematics. Ye mathematics has two faces; it is the rigorous science of Euclid but it is also something els

Mathematics presented in the Euclidean way appears as a systematic, deductive science; b
mat~~hematics in the making appears as an experimental, inductive science. Both aspects are as~~ old
the science of mathematics itself. But the second aspect is new in one respect; mathematics “in sta
nascendi,” in the process of being invented, has never before been presented in quite this manner
the student, or to the teacher himself, or to the general public.

The subject of heuristic has manifold connections; mathematicians, logicians, psychologist
educationalists, even philosophers may claim various parts of it as belonging to their special domain
The author, well aware of the possibility of criticism from opposite quarters and keenly conscious
his limitations, has one claim to make: he has some experience in solving problems and in teachin
mathematics on various levels.

The subject is more fully dealt with in a more extensive book by the author which is on the way
completion.

Stanford University, August 1, 1944

From the Preface to the Seventh Printing

I am glad to say that I have now succeeded in fulfilling, at least in part, a promise given in th preface to the first printing: The two volumes *Induction and Analogy in Mathematics* and *Patterns Plausible Inference* which constitute my recent work *Mathematics and Plausible Reasoning* continu the line of thinking begun in *How to Solve It*.

Zurich, August 30, 1954

Preface to the Second Edition

The present second edition adds, besides a few minor improvements, a new fourth part, "Problem Hints, Solutions."

As this edition was being prepared for print, a study appeared (Educational Testing Servic Princeton, N.J.; *cf. Time*, June 18, 1956) which seems to have formulated a few pertinent observatio —they are not new to the people in the know, but it was high time to formulate them for the gener public—: ". . . mathematics has the dubious honor of being the least popular subject in th curriculum . . . Future teachers pass through the elementary schools learning to dete mathematics . . . They return to the elementary school to teach a new generation to detest it."

I hope that the present edition, designed for wider diffusion, will convince some of its readers th mathematics, besides being a necessary avenue to engineering jobs and scientific knowledge, may b fun and may also open up a vista of mental activity on the highest level.

Zurich, June 30, 1956

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PART IV. PROBLEMS, HINTS, SOLUTIONS

† Contains only cross-references.

HOW TO SOLVE IT

UNDERSTANDING THE PROBLEM

First.

You have to *understand* the problem.

What is the unknown? What are the data? What is the condition? Is it possible to satisfy th
condition? Is the condition sufficient to determine the unknown? Or is it insufficient? C
redundant? Or contradictory?
Draw a figure. Introduce suitable notation.
Separate the various parts of the condition. Can you write them down?

DEVISING A PLAN

Second.

Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a *plan* of the solution.

Have you seen it before? Or have you seen the same problem in a slightly different form?
Do you know a related problem? Do you know a theorem that could be useful?
Look at the unknown! And try to think of a familiar problem having the same or a simil
unknown.
Here is a problem related to yours and solved before. Could you use it? Could you use
result? Could you use its method? Should you introduce some auxiliary element in order
make its use possible?
Could you restate the problem? Could you restate it still differently? Go back to definitions

If you cannot solve the proposed problem try to solve first some related problem. Could yo
imagine a more accessible related problem? A more general problem? A more speci
problem? An analogous problem? Could you solve a part of the problem? Keep only a pa
of the condition, drop the other part; how far is the unknown then determined, how can
vary? Could you derive something useful from the data? Could you think of other da
appropriate to determine the unknown? Could you change the unknown or the data, or bo
if necessary, so that the new unknown and the new data are nearer to each other?
Did you use all the data? Did you use the whole condition? Have you taken into account a
essential notions involved in the problem?

CARRYING OUT THE PLAN

Third.

Carry out your plan.

Carrying out your plan of the solution, *check each step*. Can you see clearly that the step
correct? Can you prove that it is correct?

LOOKING BACK

Fourth.

Examine the solution obtained.

Can you *check the result?* Can you check the argument?
Can you derive the result differently? Can you see it at a glance?
Can you use the result, or the method, for some other problem?

Foreword

by John H. Conway

How to Solve It is a wonderful book! This I realized when I first read right through it as a stude many years ago, but it has taken me a long time to appreciate just *how* wonderful it is. Why is tha One part of the answer is that the book is unique. In all my years as a student and teacher, I have nev seen another that lives up to George Polya's title by teaching you how to go about solving problem A. H. Schoenfeld correctly described its importance in his 1987 article "Polya, Problem Solving, an Education" in *Mathematics Magazine:* "For mathematics education and the world of problem solvi it marked a line of demarcation between two eras, problem solving before and after Polya."

It is one of the most successful mathematics books ever written, having sold over a million copi and been translated into seventeen languages since it first appeared in 1945. Polya later wrote tw more books about the art of doing mathematics, *Mathematics and Plausible Reasoning* (1954) an *Mathematical Discovery* (two volumes, 1962 and 1965).

The book's title makes it seem that it is directed only toward students, but in fact it is addresse just as much to their teachers. Indeed, as Polya remarks in his introduction, the first part of the boo takes the teacher's viewpoint more often than the student's.

Everybody gains that way. The student who reads the book on his own will find that overhearin Polya's comments to his non-existent teacher can bring that desirable person into being, as a imaginary but very helpful figure leaning over one's shoulder. This is what happened to me, an naturally I made heavy use of the remarks I'd found most important when I myself started teaching few years later.

But it was some time before I read the book again, and when I did, I suddenly realized that it w even more valuable than I'd thought! Many of Polya's remarks that hadn't helped me as a student no made me a better teacher of those whose problems had differed from mine. Polya had met many mo students than I had, and had obviously thought very hard about how to best help all of them lea mathematics. Perhaps his most important point is that learning must be active. As he said in a lectu on teaching, "Mathematics, you see, is not a spectator sport. To understand mathematics means to b able to do mathematics. And what does it mean [to be] doing mathematics? In the first place, it mea to be able to solve mathematical problems."

It is often said that to teach any subject well, one has to understand it "at least as well as one students do." It is a paradoxical truth that to teach mathematics well, one must also know how misunderstand it at least to the extent one's students do! If a teacher's statement can be parsed in tw or more ways, it goes without saying that some students will understand it one way and others anoth with results that can vary from the hilarious to the tragic. J. E. Littlewood gives two amusi examples of assumptions that can easily be made unconsciously and misleadingly. First, he remar that the description of the coordinate axes ("O*x* and O*y* as in 2 dimensions, O*z* vertical") in Lamb book *Mechanics* is incorrect for him, since he always worked in an armchair with his feet up! The after asking how his reader would present the picture of a closed curve lying all on one side of i tangent, he states that there are four main schools (to left or right of vertical tangent, or above below horizontal one) and that by lecturing without a figure, presuming that the curve was to the rig of its vertical tangent, he had unwittingly made nonsense for the other three schools.

I know of no better remedy for such presumptions than Polya's counsel: before trying to solve problem, the student should demonstrate his or her understanding of its statement, preferably to a re teacher, but in lieu of that, to an imagined one. Experienced mathematicians know that often th hardest part of researching a problem is understanding precisely what that problem says. They ofte

follow Polya's wise advice: "If you can't solve a problem, then there is an easier problem you can
solve: find it."

Readers who learn from this book will also want to learn about its author's life.[1]

George Polya was born György Pólya (he dropped the accents sometime later) on December 1 1887, in Budapest, Hungary, to Jakab Pólya and his wife, the former Anna Deutsch. He was baptize into the Roman Catholic faith, to which Jakab, Anna, and their three previous children, Jenő, Ilon and Flóra, had converted from Judaism in the previous year. Their fifth child, László, was born fo years later.

Jakab had changed his surname from Pollák to the more Hungarian-sounding Pólya five yea before György was born, believing that this might help him obtain a university post, which l eventually did, but only shortly before his untimely death in 1897.

At the Dániel Berzsenyi Gymnasium, György studied Greek, Latin, and German, in addition Hungarian. It is surprising to learn that there he was seemingly uninterested in mathematics, his wo in geometry deemed merely "satisfactory" compared with his "outstanding" performance in literatu geography, and other subjects. His favorite subject, outside of literature, was biology.

He enrolled at the University of Budapest in 1905, initially studying law, which he soon droppe because he found it too boring. He then obtained the certification needed to teach Latin and Hungaria at a gymnasium, a certification that he never used but of which he remained proud. Eventually h professor, Bernát Alexander, advised him that to help his studies in philosophy, he should take som mathematics and physics courses. This was how he came to mathematics. Later, he joked that l "wasn't good enough for physics, and was too good for philosophy—mathematics is in between."

In Budapest he was taught physics by Eötvös and mathematics by Fejér and was awarded doctorate after spending the academic year 1910–11 in Vienna, where he took some courses b Wirtinger and Mertens. He spent much of the next two years in Göttingen, where he met many mo mathematicians—Klein, Caratheodory, Hilbert, Runge, Landau, Weyl, Hecke, Courant, and Toeplitz- and in 1914 visited Paris, where he became acquainted with Picard and Hadamard and learned th Hurwitz had arranged an appointment for him in Zürich. He accepted this position, writing later: went to Zürich in order to be near Hurwitz, and we were in close touch for about six years, from m arrival in Zürich in 1914 to his passing [in 1919]. I was very much impressed by him and edited h works."

Of course, the First World War took place during this period. It initially had little effect on Poly who had been declared unfit for service in the Hungarian army as the result of a soccer wound. B later when the army, more desperately needing recruits, demanded that he return to fight for h country, his strong pacifist views led him to refuse. As a consequence, he was unable to visit Hunga for many years, and in fact did not do so until 1967, fifty-four years after he left.

In the meantime, he had taken Swiss citizenship and married a Swiss girl, Stella Vera Weber, i 1918. Between 1918 and 1919, he published papers on a wide range of mathematical subjects, such series, number theory, combinatorics, voting systems, astronomy, and probability. He was made extraordinary professor at the Zürich ETH in 1920, and a few years later he and Gábor Szeg published their book *Aufgaben und Lehrsatze aus der Analysis* ("Problems and Theorems Analysis"), described by G. L. Alexanderson and L. H. Lange in their obituary of Polya as mathematical masterpiece that assured their reputations."

That book appeared in 1925, after Polya had obtained a Rockefeller Fellowship to work in Englan where he collaborated with Hardy and Littlewood on what later became their book *Inequaliti* (Cambridge University Press, 1936). He used a second Rockefeller Fellowship to visit Princeto University in 1933, and while in the United States was invited by H. F. Blichfeldt to visit Stanfo

University, which he greatly enjoyed, and which ultimately became his home. Polya held ~~professorship at Stanford from 1943 until his retirement in 1953, and it was there, in 1978,~~ that l taught his last course, in combinatorics; he died on September 7, 1985, at the age of ninety-seven.

Some readers will want to know about Polya's many contributions to mathematics. Most of the relate to analysis and are too technical to be understood by non-experts, but a few are wor mentioning.

In probability theory, Polya is responsible for the now-standard term "Central Limit Theorem" an for proving that the Fourier transform of a probability measure is a characteristic function and that random walk on the integer lattice closes with probability 1 if and only if the dimension is at most 2.

In geometry, Polya independently re-enumerated the seventeen plane crystallographic groups (the first enumeration, by E. S. Fedorov, having been forgotten) and together with P. Niggli devised notation for them.

In combinatorics, Polya's Enumeration Theorem is now a standard way of counting configuratio according to their symmetry. It has been described by R. C. Read as "a remarkable theorem in remarkable paper, and a landmark in the history of combinatorial analysis."

How to Solve It was written in German during Polya's time in Zürich, which ended in 1940, whe the European situation forced him to leave for the United States. Despite the book's eventual succes four publishers rejected the English version before Princeton University Press brought it out in 194 In their hands, *How to Solve It* rapidly became—and continues to be—one of the most successf mathematical books of all time.

[1]The following biographical information is taken from that given by J. J. O'Connor and E. F. Robertson in the MacTutor Histo of Mathematics Archive (www-gap.dcs.st-and.ac.uk/~history/).

The following considerations are grouped around the preceding list of questions and suggestio entitled "How to Solve It." Any question or suggestion quoted from it will be printed in *italics*, a the whole list will be referred to simply as "the list" or as "our list."

The following pages will discuss the purpose of the list, illustrate its practical use by examples, a explain the underlying notions and mental operations. By way of preliminary explanation, this mu may be said: If, using them properly, you address these questions and suggestions to yourself, th may help you to solve your problem. If, using them properly, you address the same questions a suggestions to one of your students, you may help him to solve his problem.

The book is divided into four parts.

The title of the first part is "In the Classroom." It contains twenty sections. Each section will quoted by its number in heavy type as, for instance, "section 7." Sections 1 to 5 discuss the "Purpos of our list in general terms. Sections 6 to 17 explain what are the "Main Divisions, Main Questions" the list, and discuss a first practical example. Sections 18, 19, 20 add "More Examples."

The title of the very short second part is "How to Solve It." It is written in dialogue; a somewh idealized teacher answers short questions of a somewhat idealized student.

The third and most extensive part is a "Short Dictionary of Heuristic"; we shall refer to it as t "Dictionary." It contains sixty-seven articles arranged alphabetically. For example, the meaning of t term HEURISTIC (set in small capitals) is explained in an article with this title on page 112. When t title of such an article is referred to within the text it will be set in small capitals. Certain paragrap of a few articles are more technical; they are enclosed in square brackets. Some articles are fai closely connected with the first part to which they add further illustrations and more speci comments. Other articles go somewhat beyond the aim of the first part of which they explain t background. There is a key-article on MODERN HEURISTIC. It explains the connection of the ma articles and the plan underlying the Dictionary; it contains also directions how to find informati about particular items of the list. It must be emphasized that there is a common plan and a cert unity, because the articles of the Dictionary show the greatest outward variety. There are a few long articles devoted to the systematic though condensed discussion of some general theme; others cont more specific comments, still others cross-references, or historical data, or quotations, or aphorism or even jokes.

The Dictionary should not be read too quickly; its text is often condensed, and now and th somewhat subtle. The reader may refer to the Dictionary for information about particular points. these points come from his experience with his own problems or his own students, the reading has much better chance to be profitable.

The title of the fourth part is "Problems, Hints, Solutions." It proposes a few problems to the mo ambitious reader. Each problem is followed (in proper distance) by a "hint" that may reveal a way the result which is explained in the "solution."

We have mentioned repeatedly the "student" and the "teacher" and we shall refer to them again a again. It may be good to observe that the "student" may be a high school student, or a college stude or anyone else who is studying mathematics. Also the "teacher" may be a high school teacher, or college instructor, or anyone interested in the technique of teaching mathematics. The author looks the situation sometimes from the point of view of the student and sometimes from that of the teach (the latter case is preponderant in the first part). Yet most of the time (especially in the third part) t point of view is that of a person who is neither teacher nor student but anxious to solve the proble before him.

PURPOSE

1. Helping the student. One of the most important tasks of the teacher is to help his students. Th task is not quite easy; it demands time, practice, devotion, and sound principles.

The student should acquire as much experience of independent work as possible. But if he is le alone with his problem without any help or with insufficient help, he may make no progress at all. the teacher helps too much, nothing is left to the student. The teacher should help, but not too muc and not too little, so that the student shall have a *reasonable share of the work*.

If the student is not able to do much, the teacher should leave him at least some illusion independent work. In order to do so, the teacher should help the student discreetly, *unobtrusively*.

The best is, however, to help the student naturally. The teacher should put himself in the student place, he should see the student's case, he should try to understand what is going on in the student mind, and ask a question or indicate a step that *could have occurred to the student himself*.

2. Questions, recommendations, mental operations. Trying to help the student effectively b unobtrusively and naturally, the teacher is led to ask the same questions and to indicate the same ste again and again. Thus, in countless problems, we have to ask the question: *What is the unknown?* W may vary the words, and ask the same thing in many different ways: What is required? What do yo want to find? What are you supposed to seek? The aim of these questions is to focus the student attention upon the unknown. Sometimes, we obtain the same effect more naturally with a suggestio *Look at the unknown!* Question and suggestion aim at the same effect; they tend to provoke the san mental operation.

It seemed to the author that it might be worth while to collect and to group questions ar suggestions which are typically helpful in discussing problems with students. The list we stud contains questions and suggestions of this sort, carefully chosen and arranged; they are equally usef to the problem-solver who works by himself. If the reader is sufficiently acquainted with the list ar can see, behind the suggestion, the action suggested, he may realize that the list enumerate indirectly, *mental operations typically useful for the solution of problems*. These operations are liste in the order in which they are most likely to occur.

3. Generality is an important characteristic of the questions and suggestions contained in our lis Take the questions: *What is the unknown? What are the data? What is the condition?* These questior are generally applicable, we can ask them with good effect dealing with all sorts of problems. The use is not restricted to any subject-matter. Our problem may be algebraic or geometric, mathematic or nonmathematical, theoretical or practical, a serious problem or a mere puzzle; it makes r difference, the questions make sense and might help us to solve the problem.

There is a **restriction,** in fact, but it has nothing to do with the subject-matter. Certain questior and suggestions of the list are applicable to "problems to find" only, not to "problems to prove." If v have a problem of the latter kind we must use different questions; see PROBLEMS TO FIND, PROBLEMS T PROVE.

4. Common sense. The questions and suggestions of our list are general, but, except for the generality, they are natural, simple, obvious, and proceed from plain common sense. Take th suggestion: *Look at the unknown! And try to think of a familiar problem having the same or a simil* *unknown*. This suggestion advises you to do what you would do anyhow, without any advice, if yo ~~were seriously concerned with your problem. Are you hungry? You wish to obtain food and you~~ thin of familiar ways of obtaining food. Have you a problem of geometric construction? You wish t construct a triangle and you think of familiar ways of constructing a triangle. Have you a problem any kind? You wish to find a certain unknown, and you think of familiar ways of finding such a unknown, or some similar unknown. If you do so you follow exactly the suggestion we quoted fro our list. And you are on the right track, too; the suggestion is a good one, it suggests to you procedure which is very frequently successful.

All the questions and suggestions of our list are natural, simple, obvious, just plain common sens but they state plain common sense in general terms. They suggest a certain conduct which com naturally to any person who is seriously concerned with his problem and has some common sense. B the person who behaves the right way usually does not care to express his behavior in clear words an possibly, he cannot express it so; our list tries to express it so.

5. Teacher and student. Imitation and practice. There are two aims which the teacher may hav in view when addressing to his students a question or a suggestion of the list: First, to help the stude to solve the problem at hand. Second, to develop the student's ability so that he may solve futu problems by himself.

Experience shows that the questions and suggestions of our list, appropriately used, very frequent help the student. They have two common characteristics, common sense and generality. As the proceed from plain common sense they very often come naturally; they could have occurred to th student himself. As they are general, they help unobtrusively; they just indicate a general directio and leave plenty for the student to do.

But the two aims we mentioned before are closely connected; if the student succeeds in solving th problem at hand, he adds a little to his ability to solve problems. Then, we should not forget that o questions are general, applicable in many cases. If the same question is repeatedly helpful, the stude will scarcely fail to notice it and he will be induced to ask the question by himself in a simil situation. Asking the question repeatedly, he may succeed once in eliciting the right idea. By such success, he discovers the right way of using the question, and then he has really assimilated it.

The student may absorb a few questions of our list so well that he is finally able to put to himse the right question in the right moment and to perform the corresponding mental operation natural and vigorously. Such a student has certainly derived the greatest possible profit from our list. Wh can the teacher do in order to obtain this best possible result?

Solving problems is a practical skill like, let us say, swimming. We acquire any practical skill b imitation and practice. Trying to swim, you imitate what other people do with their hands and feet keep their heads above water, and, finally, you learn to swim by practicing swimming. Trying to solv problems, you have to observe and to imitate what other people do when solving problems an finally, you learn to do problems by doing them.

The teacher who wishes to develop his students' ability to do problems must instill some intere for problems into their minds and give them plenty of opportunity for imitation and practice. If th teacher wishes to develop in his students the mental operations which correspond to the questions an suggestions of our list, he puts these questions and suggestions to the students as often as he can do s naturally. Moreover, when the teacher solves a problem before the class, he should dramatize his ide a little and he should put to himself the same questions which he uses when helping the student Thanks to such guidance, the student will eventually discover the right use of these questions an suggestions, and doing so he will acquire something that is more important than the knowledge of an particular mathematical fact.

6. Four phases. Trying to find the solution, we may repeatedly change our point of view, our wa of looking at the problem. We have to shift our position again and again. Our conception of th problem is likely to be rather incomplete when we start the work; our outlook is different when w have made some progress; it is again different when we have almost obtained the solution.

In order to group conveniently the questions and suggestions of our list, we shall distinguish fo phases of the work. First, we have to *understand* the problem; we have to see clearly what is require Second, we have to see how the various items are connected, how the unknown is linked to the data, order to obtain the idea of the solution, to make a *plan*. Third, we *carry out* our plan. Fourth, we *lo back* at the completed solution, we review and discuss it.

Each of these phases has its importance. It may happen that a student hits upon an exceptional bright idea and jumping all preparations blurts out with the solution. Such lucky ideas, of course, a most desirable, but something very undesirable and unfortunate may result if the student leaves o any of the four phases without having a good idea. The worst may happen if the student embarks upo computations or constructions without having *understood* the problem. It is generally useless to car out details without having seen the main connection, or having made a sort of *plan*. Many mistak can be avoided if, carrying out his plan, the student *checks each step*. Some of the best effects may b lost if the student fails to reexamine and to *reconsider* the completed solution.

7. Understanding the problem. It is foolish to answer a question that you do not understand. It sad to work for an end that you do not desire. Such foolish and sad things often happen, in and out school, but the teacher should try to prevent them from happening in his class. The student shou understand the problem. But he should not only understand it, he should also desire its solution. If th student is lacking in understanding or in interest, it is not always his fault; the problem should be we chosen, not too difficult and not too easy, natural and interesting, and some time should be allowed fo natural and interesting presentation.

First of all, the verbal statement of the problem must be understood. The teacher can check this, u to a certain extent; he asks the student to repeat the statement, and the student should be able to sta the problem fluently. The student should also be able to point out the principal parts of the problem the unknown, the data, the condition. Hence, the teacher can seldom afford to miss the question *What is the unknown? What are the data? What is the condition?*

The student should consider the principal parts of the problem attentively, repeatedly, and fro various sides. If there is a figure connected with the problem he should *draw a figure* and point out o it the unknown and the data. If it is necessary to give names to these objects he should *introdu suitable notation;* devoting some attention to the appropriate choice of signs, he is obliged to consid the objects for which the signs have to be chosen. There is another question which may be useful this preparatory stage provided that we do not expect a definitive answer but just a provisional answe a guess: *Is it possible to satisfy the condition?*

(In the exposition of Part II [p. 33] “Understanding the problem” is subdivided into two stage “Getting acquainted” and “Working for better understanding.”)

8. Example. Let us illustrate some of the points explained in the foregoing section. We take th following simple problem: *Find the diagonal of a rectangular parallelepiped of which the length, th width, and the height are known.*

In order to discuss this problem profitably, the students must be familiar with the theorem o Pythagoras, and with some of its applications in plane geometry, but they may have very litt systematic knowledge in solid geometry. The teacher may rely here upon the student’s unsophisticate familiarity with spatial relations.

The teacher can make the problem interesting by making it concrete. The classroom is rec~~tangular parallelepiped whose dimensions could be measured, and can be estimated; the~~ studen have to find, to “measure indirectly,” the diagonal of the classroom. The teacher points out the lengt the width, and the height of the classroom, indicates the diagonal with a gesture, and enlivens h figure, drawn on the blackboard, by referring repeatedly to the classroom.

The dialogue between the teacher and the students may start as follows:

“What is the unknown?”

“The length of the diagonal of a parallelepiped.”

“What are the data?”

“The length, the width, and the height of the parallelepiped.”

“Introduce suitable notation. Which letter should denote the unknown?”

“x.”

“Which letters would you choose for the length, the width, and the height?”

“a, b, c.”

“What is the condition, linking *a, b, c*, and *x?”*

“x is the diagonal of the parallelepiped of which *a, b*, and *c* are the length, the width, and t height.”

“Is it a reasonable problem? I mean, *is the condition sufficient to determine the unknown?”*

“Yes, it is. If we know *a, b, c*, we know the parallelepiped. If the parallelepiped is determined, t diagonal is determined.”

9. Devising a plan. We have a plan when we know, or know at least in outline, which calculation computations, or constructions we have to perform in order to obtain the unknown. The way fro understanding the problem to conceiving a plan may be long and tortuous. In fact, the ma achievement in the solution of a problem is to conceive the idea of a plan. This idea may emer gradually. Or, after apparently unsuccessful trials and a period of hesitation, it may occur suddenly, i a flash, as a “bright idea.” The best that the teacher can do for the student is to procure for him, b unobtrusive help, a bright idea. The questions and suggestions we are going to discuss tend to provol such an idea.

In order to be able to see the student’s position, the teacher should think of his own experience, his difficulties and successes in solving problems.

We know, of course, that it is hard to have a good idea if we have little knowledge of the subjec and impossible to have it if we have no knowledge. Good ideas are based on past experience a formerly acquired knowledge. Mere remembering is not enough for a good idea, but we cannot ha any good idea without recollecting some pertinent facts; materials alone are not enough f constructing a house but we cannot construct a house without collecting the necessary materials. T materials necessary for solving a mathematical problem are certain relevant items of our former acquired mathematical knowledge, as formerly solved problems, or formerly proved theorems. Thu it is often appropriate to start the work with the question: *Do you know a related problem?*

The difficulty is that there are usually too many problems which are somewhat related to o present problem, that is, have some point in common with it. How can we choose the one, or the fe which are really useful? There is a suggestion that puts our finger on an essential common point: *Lo at the unknown! And try to think of a familiar problem having the same or a similar unknown.*

If we succeed in recalling a formerly solved problem which is closely related to our prese problem, we are lucky. We should try to deserve such luck; we may deserve it by exploiting it. *Here a problem related to yours and solved before. Could you use it?*

The foregoing questions, well understood and seriously considered, very often help to start the rig train of ideas; but they cannot help always, they cannot work magic. If they do not work, we must loo aroun~~d for some other appropriate point of contact, and explore the various aspects of our proble~~m; w have to vary, to transform, to modify the problem. *Could you restate the problem?* Some of th questions of our list hint specific means to vary the problem, as generalization, specialization, use analogy, dropping a part of the condition, and so on; the details are important but we cannot go in them now. Variation of the problem may lead to some appropriate auxiliary problem: *If you cann solve the proposed problem try to solve first some related problem.*

Trying to apply various known problems or theorems, considering various modification experimenting with various auxiliary problems, we may stray so far from our original problem that w are in danger of losing it altogether. Yet there is a good question that may bring us back to it: *Did yo use all the data? Did you use the whole condition?*

10. Example. We return to the example considered in section 8. As we left it, the students ju succeeded in understanding the problem and showed some mild interest in it. They could now hav some ideas of their own, some initiative. If the teacher, having watched sharply, cannot detect an sign of such initiative he has to resume carefully his dialogue with the students. He must be prepare to repeat with some modification the questions which the students do not answer. He must be prepare to meet often with the disconcerting silence of the students (which will be indicated by dots . . . . .).

"Do you know a related problem?"

. . . . .

"Look at the unknown! Do you know a problem having the same unknown?"

. . . . .

"Well, *what is the unknown?"*

"The diagonal of a parallelepiped."

"Do you know any *problem with the same unknown?"*

"No. We have not had any problem yet about the diagonal of a parallelepiped."

"Do you know any *problem with a similar unknown?"*

. . . . .

"You see, the diagonal is a segment, the segment of a straight line. Did you never solve a proble whose unknown was the length of a line?"

"Of course, we have solved such problems. For instance, to find a side of a right triangle."

"Good! *Here is a problem related to yours and solved before. Could you use it?"*

. . . . .

"You were lucky enough to remember a problem which is related to your present one and whic you solved before. Would you like to use it? *Could you introduce some auxiliary element in order t make its use possible?"*

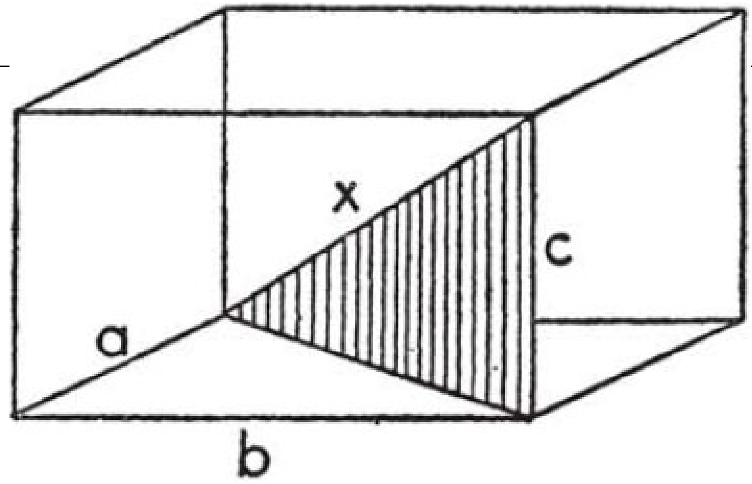

FIG. 1

. . . . .

“Look here, the problem you remembered is about a triangle. Have you any triangle in yo figure?”

Let us hope that the last hint was explicit enough to provoke the idea of the solution which is introduce a right triangle, (emphasized in Fig. 1) of which the required diagonal is the hypotenuse. Y the teacher should be prepared for the case that even this fairly explicit hint is insufficient to shake t torpor of the students; and so he should be prepared to use a whole gamut of more and more explic hints.

“Would you like to have a triangle in the figure?”

“What sort of triangle would you like to have in the figure?”

“You cannot find yet the diagonal; but you said that you could find the side of a triangle. Nov what will you do?”

“Could you find the diagonal, if it were a side of a triangle?”

When, eventually, with more or less help, the students succeed in introducing the decisive auxilia element, the right triangle emphasized in Fig. 1, the teacher should convince himself that the studen see sufficiently far ahead before encouraging them to go into actual calculations.

“I think that it was a good idea to draw that triangle. You have now a triangle; but have you t unknown?”

“The unknown is the hypotenuse of the triangle; we can calculate it by the theorem of Pythagoras.

“You can, if both legs are known; but are they?”

“One leg is given, it is *c*. And the other, I think, is not difficult to find. Yes, the other leg is th hypotenuse of another right triangle.”

“Very good! Now I see that you have a plan.”

11. Carrying out the plan. To devise a plan, to conceive the idea of the solution is not easy. takes so much to succeed; formerly acquired knowledge, good mental habits, concentration upon t purpose, and one more thing: good luck. To carry out the plan is much easier; what we need is main patience.

The plan gives a general outline; we have to convince ourselves that the details fit into the outlin and so we have to examine the details one after the other, patiently, till everything is perfectly clea and no obscure corner remains in which an error could be hidden.

If the student has really conceived a plan, the teacher has now a relatively peaceful time. The ma danger is that the student forgets his plan. This may easily happen if the student received his pl from outside, and accepted it on the authority of the teacher; but if he worked for it himself, even wi