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T. Y. Lam

Lectures on Modules and Rings

With 43 Figures

Springer

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To Chee King

Juwen, Fumei, Juleen, Tsai Yu

Preface

Textbook writing must be one of the cruelest of self-inflicted tortures.

– Carl Faith
Math Reviews 54: 5281

So why didn't I heed the warning of a wise colleague, especially one who is a great expert in the subject of modules and rings? The answer is simple: I did not learn about it until it was too late!

My writing project in ring theory started in 1983 after I taught a year-long course in the subject at Berkeley. My original plan was to write up my lectures and publish them as a graduate text in a couple of years. My hopes of carrying out this plan on schedule were, however, quickly dashed as I began to realize how much material was at hand and how little time I had at my disposal. As the years went by, I added further material to my notes, and used them to teach different versions of the course. Eventually, I came to the realization that writing a single volume would not fully accomplish my original goal of giving a comprehensive treatment of basic ring theory.

At the suggestion of Ulrike Schmickler-Hirzebruch, then Mathematics Editor of Springer-Verlag, I completed the first part of my project and published the write-up in 1991 as *A First Course in Noncommutative Rings*, GTM 131, hereafter referred to as *First Course* (or simply *FC*). This volume contained a treatment of the Wedderburn-Artin theory of semisimple rings, Jacobson's theory of the radical, representation theory of groups and algebras, prime and semiprime rings, division rings, ordered rings, local and semilocal rings, culminating in the theory of perfect and semiperfect rings. The publication of this volume was accompanied several years later by that of *Exercises in Classical Ring Theory*, which contained full solutions of (and additional commentary on) all exercises in *FC*. For further topics in ring theory not yet treated in *FC*, the reader was referred to a forthcoming second volume, which, for lack of a better name, was tentatively billed as *A Second Course in Noncommutative Rings*.

One primary subject matter I had in mind for the second volume was that part of ring theory in which the consideration of modules plays a crucial role. While an early chapter of *FC* on representation theory dealt with modules over finite-dimensional algebras (such as group algebras of finite groups over fields), the theory of modules over more general rings did not receive a full treatment in that text. This second volume, therefore, begins with the theory of special classes of modules (free, projective, injective, and flat modules) and the theory of homological dimensions of modules and rings. This material occupies the first two chapters. We then go on to present, in Chapter 3, the theory of uniform dimensions, complements, singular submodules and rational hulls; here, the notions of essentiality and denseness of submodules play a key role. In this chapter, we also encounter several interesting classes of rings, notably Rickart rings and Baer rings, Johnson's nonsingular rings, and Kasch rings, not to mention the hereditary and semihereditary rings that have already figured in the first two chapters.

Another important topic in classical ring theory not yet treated in *FC* was the theory of rings of quotients. This topic is taken up in Chapter 4 of the present text, in which we present Ore's theory of noncommutative localization, followed by a treatment of Goldie's all-important theorem characterizing semiprime right Goldie rings as right orders in semisimple rings. The latter theorem, truly a landmark in ring theory, brought the subject into its modern age, and laid new firm foundations for the theory of noncommutative noetherian rings. Another closely allied theory is that of maximal rings of quotients, due to Findlay, Lambek and Utumi. This theory has a universal appeal, since every ring has a maximal (left, right) ring of quotients. Chapter 5 develops this theory, taking full advantage of the material on injective and rational hulls of modules presented in the previous chapters. In this theory, the theorems of Johnson and Gabriel characterizing rings whose maximal right rings of quotients are von Neumann regular or semisimple may be viewed as analogues of Goldie's theorem mentioned earlier.

One theme that runs like a red thread through Chapters 1–5 is that of self-injective rings. The noetherian self-injective rings, commonly known as quasi-Frobenius (or QF) rings, occupy an especially important place in ring theory. Group algebras of finite groups provided the earliest nontrivial examples of QF rings; in fact, they are examples of finite-dimensional Frobenius algebras that were studied already in the first chapter. The general theory of Frobenius and quasi-Frobenius rings is developed in considerable detail in Chapter 6. Over such rings, we witness a remarkable "perfect duality" between finitely generated left and right modules. Much of the beautiful mathematics here goes back to Dieudonné, Nakayama, Nesbitt, Brauer, and Frobenius. This theory serves eventually as the model for the general theory of duality between module categories developed by Kiiti Morita in his classical paper in 1958. Our text concludes with an exposition, in Chapter 7, of this duality theory, along with the equally significant theory of module category equivalences developed concomitantly by Kiiti Morita.

Although the present text was originally conceived as a sequel to *FC*, the material covered here is largely independent of that in *First Course*, and can be used as a text in its own right for a course in ring theory stressing the role of modules over rings. In fact, I have myself used the material in this manner in a couple of courses at Berkeley. For this reason, it is deemed appropriate to rename the book so as to decouple it from *First Course*; hence the present title, *Lectures on Modules and Rings*. I am fully conscious of the fact that this title is a permutation of *Lectures on Rings and Modules* by Lambek — and even more conscious of the fact that my name happens to be a subset of his!

For readers using this textbook without having read *FC*, some orienting remarks are in order. While it is true that, in various places, references are made to *First Course*, these references are mostly for really basic material in ring theory, such as the Wedderburn-Artin Theorem, facts about the Jacobson radical, noetherian and artinian rings, local and semilocal rings, or the like. These are topics that a graduate student is likely to have learned from a good first-year graduate course in algebra using a strong text such as that of Lang, Hungerford, or Isaacs. For a student with this kind of background, the present text can be used largely independently of *FC*. For others, an occasional consultation with *FC*, together with a willingness to take some ring-theoretic facts for granted, should be enough to help them navigate through the present text with ease. The *Notes to the Reader* section following the Table of Contents spells out in detail some of the things, mathematical or otherwise, which will be useful to know in working with this text. For the reader's convenience, we have also included a fairly complete list of the notations used in the book, together with a partial list of frequently used abbreviations.

In writing the present text, I was guided by three basic principles. First, I tried to write in the way I give my lectures. This means I took it upon myself to select the most central topics to be taught, and I tried to expound these topics by using the clearest and most efficient approach possible, without the hindrance of heavy machinery or undue abstractions. As a result, all material in the text should be well-suited for direct class presentations. Second, I put a premium on the use of examples. Modules and rings are truly ubiquitous objects, and they are a delight to construct. Yet, a number of current ring theory books were almost totally devoid of examples. To reverse this trend, we did it with a vengeance: an abundance of examples was offered virtually every step of the way, to illustrate everything from concepts, definitions, to theorems. It is hoped that the unusual number of examples included in this text makes it fun to read. Third, I recognized the vital role of problem solving in the learning process. Thus, I have made a special effort to compile extensive sets of exercises for all sections of the book. Varying from the routine to the most challenging, the compendium of (exactly) 600 exercises greatly extends the scope of the text, and offers a rich additional source of information to novices and experts alike. Also, to maintain a good control over the quality and propriety of these exercises, I made it a point to do each and every one of them myself. Solutions to all exercises in this text, with additional commentary on the majority of them, will hopefully appear later in the form of a separate problem book.

As I came to the end of my arduous writing journey that began as early as 1983, I grimaced over the one-liner of Carl Faith quoted at the beginning of this preface. Torture it no doubt was, and the irony lay indeed in the fact that I had chosen to inflict it upon myself. But surely every author had a compelling reason for writing his or her opus; the labor and pain, however excruciating, were only a part of the price to pay for the joyful creation of a new brain-child!

If I had any regrets about this volume, it would only be that I did not find it possible to treat all of the interesting ring-theoretic topics that I would have liked to include. Among the most glaring omissions are: the dimension theory and torsion theory of rings, noncommutative noetherian rings and PI rings, and the theory of central simple algebras and enveloping algebras. Some of these topics were "promised" in *FC*, but obviously, to treat any of them would have further increased the size of this book. I still fondly remember that, in Professor G.-C. Rota's humorous review of my *First Course*, he mused over some mathematicians' unforgiving (and often vociferous) reactions to omissions of their favorite results in textbooks, and gave the example of a "Professor Neanderthal of Redwood Poly", who, upon seeing my book, was confirmed in his darkest suspicions that I had failed to "include a mention, let alone a proof, of the Worpitzky-Yamamoto Theorem." Sadly enough, to the Professor Neanderthals of the world, I must shamefully confess that, even in this *second* volume in noncommutative ring theory, I still did not manage to include a mention, let alone a proof, of that omnipotent Worpitzky-Yamamoto Theorem!

Obviously, a book like this could not have been written without the generous help of many others. First, I thank the audiences in several of the ring theory courses I taught at Berkeley in the last 15 years. While it is not possible to name them all, I note that the many talented (former) students who attended my classes included Ka Hin Leung, Tara Smith, David Moulton, Bjorn Poonen, Arthur Drisko, Peter Farbman, Geir Agnarsson, Ioannis Emmanouil, Daniel Isaksen, Romyar Sharifi, Nghi Nguyen, Greg Marks, Will Murray, and Monica Vazirani. They have corrected a number of mistakes in my presentations, and their many pertinent questions and remarks in class have led to various improvements in the text. I also thank heartily all those who have read portions of preliminary versions of the book and offered corrections, suggestions, and other constructive comments. This includes Ioannis Emmanouil, Greg Marks, Will Murray, Monica Vazirani, Scott Annin, Stefan Schmidt, André Leroy, S. K. Jain, Charles Curtis, Rad Dimitrić, Ellen Kirkman, and Dan Shapiro. Other colleagues helped by providing proofs, examples and counterexamples, suggesting exercises, pointing out references, or answering my mathematical queries: among them, I should especially thank George Bergman, Hendrik Lenstra, Jr., Carl Faith, Barbara Osofsky, Lance Small, Susan Montgomery, Joseph Rotman, Richard Swan, David Eisenbud, Craig Huneke, and Birge Huisgen-Zimmermann.

Last, first, and always, I owe the greatest debt to members of my family. At the risk of sounding like a broken record, I must once more thank my wife Chee-King for graciously enduring yet another book project. She can now take comfort in my solemn pledge that there will *not* be a *Third Course*! The company of our four children brings cheers and joy into my life, which keep me going. I thank them fondly for their love, devotion and unstinting support.

Berkeley, California T.Y.L.
July 4, 1998

Contents

Notes to the Reader

This book consists of nineteen sections (§§1–19), which, for ease of reference, are numbered consecutively, independently of the seven chapters. Thus, a cross-reference such as (12.7) refers to the result (lemma, theorem, example, or remark) so labeled in §12. On the other hand, Exercise (12.7) will refer to Exercise 7 in the exercise set appearing at the end of §12. In referring to an exercise appearing (or to appear) in the same section, we shall sometimes drop the section number from the reference. Thus, when we refer to "Exercise 7" *within* §12, we shall mean Exercise (12.7). A reference in brackets, such as Amitsur [72] (or [Amitsur: 72]) shall refer to the 1972 paper/book of Amitsur listed in the reference section at the end of the text.

Throughout the text, some familiarity with elementary ring theory is assumed, so that we can start our discussion at an "intermediate" level. Most (if not all) of the facts we need from commutative and noncommutative ring theory are available from standard first-year graduate algebra texts such as those of Lang, Hungerford, and Isaacs, and certainly from the author's *First Course in Noncommutative Rings* (GTM 131). The latter work will be referred to throughout as *First Course* (or simply *FC*). For the reader's convenience, we summarize below a number of basic ring-theoretic notions and results which will prove to be handy in working with the text.

Unless otherwise stated, a ring R means a ring with an identity element 1, and a subring of R means a subring $S \subseteq R$ with $1 \in S$. The word "ideal" always means a two-sided ideal; an adjective such as "noetherian" likewise means right and left noetherian. A ring homomorphism from R to R' is supposed to take the identity of R to that of R'. Left and right R-modules are always assumed to be unital; homomorphisms between modules are usually written (and composed) on the opposite side of scalars. "Semisimple rings" are in the sense of Wedderburn, Noether and Artin: these are rings that are semisimple as left (right) modules over themselves. We shall use freely the classical Wedderburn-Artin Theorem (*FC* (3.5)), which states that a ring R is semisimple iff it is isomorphic to a direct product $\mathbb{M}_{n_1}(D_1) \times \cdots \times \mathbb{M}_{n_r}(D_r)$, where the D_i's are division rings. The $\mathbb{M}_{n_i}(D_i)$'s are called the *simple components* of R; these are the most typical simple artinian rings. A classical theorem of Maschke states that the group algebra kG of a finite group G over a field k of characteristic prime to $|G|$ is semisimple.

The *Jacobson radical* of a ring R, denoted by rad R, is the intersection of the maximal left (right) ideals of R; its elements are exactly those which act trivially on all left (right) R-modules. If rad $R = 0$, R is said to be *Jacobson semisimple* (or just *J-semisimple*). Such rings generalize the classical semisimple rings, in that semisimple rings are precisely the *artinian* J-semisimple rings. A ring R is called *semilocal* if $R/\text{rad } R$ is artinian (and hence semisimple); in the case when R is commutative, this amounts to R having only a finite number of maximal ideals. If R is semilocal and rad R is nilpotent, R is said to be *semiprimary*. Over such a ring, the Hopkins-Levitzki Theorem (*FC*-(4.15)) states that any noetherian module has a composition series. This theorem implies that left (right) artinian rings are precisely the semiprimary left (right) noetherian rings.

In a ring R, a *prime ideal* is an ideal $\mathfrak{p} \subsetneq R$ such that $aRb \subseteq \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$; a *semiprime ideal* is an ideal $\mathfrak{C}$ such that $aRa \subseteq \mathfrak{C}$ implies $a \in \mathfrak{C}$. Semiprime ideals are exactly intersections of prime ideals. A ring R is called *prime* (*semiprime*) if the zero ideal is prime (semiprime). The *prime radical* (a.k.a. Baer radical, or lower nilradical[1]) of a ring R is denoted by Nil_*R: it is the smallest semiprime ideal of R (given by the intersection of all of its prime ideals). Thus, R is semiprime iff $\mathrm{Nil}_*R = 0$, iff R has no nonzero nilpotent ideals. In case R is commutative, Nil_*R is just $\mathrm{Nil}(R)$, the set of all nilpotent elements in R; R being semiprime in this case simply means that R is a *reduced ring*, that is, a ring without nonzero nilpotent elements. In general, $\mathrm{Nil}_*R \subseteq \text{rad } R$, with equality in case R is a 1-sided artinian ring.

A *domain* is a nonzero ring in which there is no 0-divisor (other than 0). Domains are prime rings, and reduced rings are semiprime rings. A *local ring* is a ring R in which there is a unique maximal left (right) ideal $\mathfrak{m}$; in this case, we often say that $(R, \mathfrak{m})$ is a local ring. For such rings, rad $R = \mathfrak{m}$, and $R/\text{rad } R$ is a division ring. An element a in a ring R is called *regular* if it is neither a left nor a right 0-divisor, and *von Neumann regular* if $a \in aRa$. The ring R itself is called von Neumann regular if every $a \in R$ is von Neumann regular. Such rings are characterized by the fact that every principal (resp., finitely generated) left ideal is generated by an idempotent element.

A nonzero module M is said to be *simple* if it has no submodules other than (0) and M, and *indecomposable* if it is not a direct sum of two nonzero submodules. The *socle* of a module M, denoted by $\mathrm{soc}(M)$, is the sum of all simple submodules of M. In case M is R_R (R viewed as a right module over itself), the socle is always an ideal of R, and is given by the left annihilator of rad R if R is 1-sided artinian (*FC*-Exer. (4.20)). In general, however, $\mathrm{soc}(R_R) \neq \mathrm{soc}(_RR)$.

[1] The upper nilradical Nil^*R (the largest nil ideal in R) will not be needed in this book.

Partial List of Notations

$\mathbb{Z}$	ring of integers
$\mathbb{Q}$	field of rational numbers
$\mathbb{R}$	field of real numbers
$\mathbb{C}$	field of complex numbers
$\mathbb{F}_q$	finite field with q elements
$\mathbb{Z}_n$, C_n	the cyclic group $\mathbb{Z}/n\mathbb{Z}$
C_{p^∞}	the Prüfer p-group
$\emptyset$	the empty set
$\subseteq$, $\subset$	used interchangeably for inclusion
$\subsetneq$	strict inclusion
$\lvert A\rvert$, Card A	used interchangeably for the cardinality of the set A
$A \setminus B$	set-theoretic difference
$A \hookrightarrow B$	injective mapping from A into B
$A \twoheadrightarrow B$	surjective mapping from A onto B
δ_{ij}	Kronecker deltas
E_{ij}	standard matrix units
M^t, M^T	transpose of the matrix M
$\mathbb{M}_n(S)$	set of $n \times n$ matrices with entries from S
$\mathrm{GL}_n(S)$	group of invertible $n \times n$ matrices over S
$\mathrm{GL}(V)$	group of linear automorphisms of a vector space V
$Z(G)$	center of the group (or the ring) G
$C_G(A)$	centralizer of A in G
$[G : H]$	index of subgroup H in a group G
$[K : F]$	field extension degree
$\mathfrak{M}_R$, ${}_R\mathfrak{M}$	category of right (left) R-modules
$\mathfrak{M}_R^{fg}$, ${}_R^{fg}\mathfrak{M}$	category of f.g. right (left) R-modules
M_R, ${}_RN$	right R-module M, left R-module N
${}_RM_S$	(R, S)-bimodule M
$M \otimes_R N$	tensor product of M_R and ${}_RN$
$\mathrm{Hom}_R(M, N)$	group of R-homomorphisms from M to N
$\mathrm{End}_R(M)$	ring of R-endomorphisms of M

nM	$M \oplus \cdots \oplus M$ (n times)
$M^{(I)}$	$\sum_{i \in I} M$ (direct sum of I copies of M)
M^I	$\prod_{i \in I} M$ (direct product of I copies of M)
$\Lambda^n(M)$	n-th exterior power of M
$\mathrm{soc}(M)$	socle of M
$\mathrm{rad}(M)$	radical of M
$\mathrm{Ass}(M)$	set of associated primes of M
$E(M)$	injective hull (or envelope) of M
$\tilde{E}(M)$	rational hull (or completion) of M
$Z(M)$	singular submodule of M
length M	(composition) length of M
u.dim M	uniform dimension of M
rank M	torsion-free rank or (Goldie) reduced rank of M
$\rho(M)$, $\rho_R(M)$	ρ-rank of M_R
M^*	R-dual of an R-module M
M', $M^{\wedge}$	character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M_R
$\hat{M}$, $M^{\wedge}$	k-dual of a k-vector space (or k-algebra) M
$N^{c\ell}$, $\mathrm{cl}(N)$	Goldie closure of a submodule $N \subseteq M$
$N \subseteq_e M$	N is an essential submodule of M
$N \subseteq_d M$	N is a dense submodule of M
$N \subseteq_c M$	N is a complement submodule (or closed submodule) of M
R^{op}	the opposite ring of R
$\mathrm{U}(R)$, R^*	group of units of the ring R
$\mathrm{U}(D)$, D^*, $\dot{D}$	multiplicative group of the division ring D
$\mathcal{C}_R$	set of regular elements of a ring R
$\mathcal{C}(N)$	set of elements which are regular modulo the ideal N
rad R	Jacobson radical of R
$\mathrm{Nil}^* R$	upper nilradical of R
$\mathrm{Nil}_* R$	lower nilradical (a.k.a. prime radical) of R
$\mathrm{Nil}(R)$	nilradical of a commutative ring R
$A^\ell(R)$, $A^r(R)$	left, right artinian radical of R
$\mathrm{Max}(R)$	set of maximal ideals of a ring R
$\mathrm{Spec}(R)$	set of prime ideals of a ring R
$\Omega(R)$	set of isomorphism classes of indecomposable injective modules over R
$\mathrm{soc}(R_R)$, $\mathrm{soc}(_R R)$	right (left) socle of R
$Z(R_R)$, $Z(_R R)$	right (left) singular ideal of R
$\mathrm{Pic}(R)$	Picard group of a commutative ring R
R_S	universal S-inverting ring for R
RS^{-1}, $S^{-1}R$	right (left) Ore localization of R at S
$R_{\mathfrak{p}}$	localization of (commutative) R at prime ideal $\mathfrak{p}$
$Q^r_{cl}(R)$, $Q^\ell_{cl}(R)$	classical right (left) ring of quotients for R
$Q_{cl}(R)$, $Q(R)$	the above when R is commutative

$Q^r_{\max}(R)$, $Q^\ell_{\max}(R)$	maximal right (left) ring of quotients for R
$Q^r(R)$, $Q^\ell(R)$	Martindale right (left) ring of quotients
$Q^s(R)$	symmetric Martindale ring of quotients
$\mathrm{ann}_r(S)$, $\mathrm{ann}_\ell(S)$	right, left annihilators of the set S
$\mathrm{ann}^M(S)$	annihilator of S taken in M
$\varinjlim$	injective (or direct) limit
$\varprojlim$	projective (or inverse) limit
kG, $k[G]$	(semi)group ring of the (semi)group G over the ring k
$k[x_i : i \in I]$	polynomial ring over k with (commuting) variables $\{x_i : i \in I\}$
$k\langle x_i : i \in I\rangle$	free ring over k generated by $\{x_i : i \in I\}$
$k[[x_1, \ldots, x_n]]$	power series in the x_i's over k

Partial List of Abbreviations

FC	*First Course in Noncommutative Rings*
RHS, LHS	right-hand side, left-hand side
ACC	ascending chain condition
DCC	descending chain condition
IBN	"Invariant Basis Number" property
PRIR, PRID	principal right ideal ring (domain)
PLIR, PLID	principal left ideal ring (domain)
FFR	finite free resolution
QF	quasi-Frobenius
PF	pseudo-Frobenius
PP	"principal implies projective"
PI	"polynomial identity" (ring, algebra)
CS	"closed submodules are summands"
QI	quasi-injective (module)
Obj	object(s) (of a category)
iff	if and only if
resp.	respectively
ker	kernel
coker	cokernel
im	image
f.cog.	finitely cogenerated
f.g.	finitely generated
f.p.	finitely presented
f.r.	finitely related
l.c.	linearly compact
pd	projective dimension
id	injective dimension
fd	flat dimension
wd	weak dimension (of a ring)
r.gl.dim	right global dimension (of a ring)
l.gl.dim	left global dimension (of a ring)

Chapter 1

Free Modules, Projective, and Injective Modules

An effective way to understand the behavior of a ring R is to study the various ways in which R acts on its left and right modules. Thus, the theory of modules can be expected to be an essential chapter in the theory of rings. Classically, modules were used in the study of representation theory (see Chapter 3 in *First Course*). With the advent of homological methods in the 1950s, the theory of modules has become much broader in scope. Nowadays, this theory is often pursued as an end in itself. Quite a few books have been written on the theory of modules alone.

This chapter and the next are entirely devoted to module theory, with emphasis on the *homological* viewpoint. In the three sections of this chapter, we give an introduction to the notions of *freeness*, *projectivity* and *injectivity* for (right) modules. Flatness and homological dimensions will be taken up in the next chapter. The material in these two chapters constitutes the backbone of the modern homological theory of modules.

Limitation of space has made it necessary for us to present only the basic facts and the most standard theorems on free, projective, and injective modules in this chapter. Nevertheless, we will be able to introduce the reader to a number of interesting results. Readers desiring further reading in these areas are encouraged to consult the monographs of Faith [76], Kasch [82], Anderson-Fuller [92], and Wisbauer [91].

Much of the material in this chapter will be needed in a fundamental way in the subsequent chapters. For instance, both projectives and injectives will play a role in the study of flat modules, and are vital for the theory of homological dimensions in the next chapter. The idea of essential extensions will prove to be indispensable (even essential!) in dealing with uniform dimensions and complements in Chapter 3, and the formation of the injective hull of a ring is crucial for the theory of rings of quotients to be developed in Chapters 4 and 5. Finally, projective and injective modules are exactly what we need in Chapter 7 in studying Morita's important theory of equivalences and dualities for categories of modules over rings. Given the key roles projective and injective modules play in this book, the reader will be well-advised to study this beginning chapter carefully. However, the three sections in this chapter are largely independent, and can be tackled "almost" in any order.

Thus, readers interested in a quick start on projective (resp. injective) modules can proceed directly to §2 (resp. §3), and return to §1 whenever they please.

§1. Free Modules

§1A. *Invariant Basis Number (IBN)*

For a given ring R, we write $\mathfrak{M}_R$ (resp. ${}_R\mathfrak{M}$) for the category of right (resp. left) R-modules. The notation M_R (resp. ${}_RN$) means that M (resp. N) is a given right (resp. left) R-module. We shall also indicate this sometimes by writing $M \in \mathfrak{M}_R$, although strictly speaking we should have written $M \in \mathrm{Obj}(\mathfrak{M}_R)$ since M is an object in (and not a member of) $\mathfrak{M}_R$. Throughout this chapter, we work with right modules, and write homomorphisms on the left so that we use the usual left-hand rule for the composition of homomorphisms. It goes without saying that all results have analogues for left modules (for which the homomorphisms are written on the right).

We begin our discussion by treating free modules in §1. For any ring R, the module R_R is called the *right regular module*. A right module F_R is called *free* if it is isomorphic to a (possibly infinite) direct sum of copies of R_R. We write $R^{(I)}$ for the direct sum $\bigoplus_{i \in I} R_i$ where each R_i is a copy of R_R, and I is an arbitrary indexing set. The notation R^I will be reserved for the direct product $\prod_{i \in I} R_i$. If I is a *finite* set with n elements, then the direct sum and the direct product coincide; in this case we write R^n for $R^{(I)} = R^I$.

There are two more ways of describing a free module, with which we assume the reader is familiar. First, a module F_R is free iff it has a *basis*, i.e. a set $\{e_i : i \in I\} \subseteq F$ such that any element of F is a unique finite (right) linear combination of the e_i's. Second, a module F_R with a subset $B = \{e_i : i \in I\}$ is free with B as a basis iff the following "universal property" holds: for any family of elements $\{m_i : i \in I\}$ in any $M \in \mathfrak{M}_R$, there is a unique R-homomorphism $f : F \to M$ with $f(e_i) = m_i$ for all $i \in I$. By convention, the zero module (0) is free with the empty set $\emptyset$ as basis.

As an example, note that free $\mathbb{Z}$-modules are just the free abelian groups. If R is a division ring, then all $M \in \mathfrak{M}_R$ are free and the usual facts from linear algebra on independent sets and generating sets in vector spaces are valid. However, over general rings, many of these facts may no longer hold. One fact that does hold over any ring R is the following.

(1.1) Generation Lemma. *Let $\{e_i : i \in I\}$ be a minimal generating set of $M \in \mathfrak{M}_R$ where the cardinality $|I|$ is infinite. Then M cannot be generated by fewer than $|I|$ elements.*

Proof. Consider any set $A = \{a_j : j \in J\} \subseteq M$ where $|J| < |I|$. Each a_j is in the span of a finite number of the e_i's. First assume $|J|$ is infinite. Then there exists a subset $I_0 \subseteq I$ with $|I_0| \leq |J| \cdot \aleph_0 = |J|$ such that each a_j is in the span of $\{e_i : i \in I_0\}$. Since $|I_0| \le |J| < |I|$, we have

$$\mathrm{span}(A) \subseteq \mathrm{span}\{e_i : i \in I_0\} \subsetneq M,$$

as desired. If $|J|$ is finite, then $\mathrm{span}(A)$ is contained in the span of a finite number of the e_i's. Since $|I|$ is infinite, the latter span is again properly contained in M. $\square$

Remark. As the reader can see, the preceding proof already works under the weaker hypothesis that (I is infinite and) no subset $\{e_i : i \in I_0\}$ of $\{e_i : i \in I\}$ with $|I_0| < |I|$ can generate M.

From this Lemma, we can check easily that "finitely generated free module" is synonymous with "R^n for some non-negative integer n". More importantly, the Generation Lemma has the following interesting consequence.

(1.2) Corollary. *If $R^{(I)} \cong R^{(J)}$ as right R-modules, where $R \neq (0)$ and I is infinite, then $|I| = |J|$. (The rank of $R^{(I)}$, taken to be the cardinal $|I|$, is therefore well-defined in this case.)*

If I, J are both *finite* sets, this Corollary may no longer hold, as we shall see below. This prompts the following definition.

(1.3) Definition. A ring R is said to have (right) IBN ("*Invariant Basis Number*") if, for any natural numbers n, m, $R^n \cong R^m$ (as right modules) implies that $n = m$. Note that this means that any two bases on a f.g.[1] free module F_R have the same (finite) number of elements. This common number is defined to be the *rank* of F.

Another shorthand occasionally used for "IBN" in the literature is "URP", for "Unique Rank Property". As aptly pointed out by D. Shapiro, "URP" has the advantage of being more pronounceable (it rhymes with "burp"). In this book, we shall follow the majority of ring theorists and use the more traditional (if unpronounceable) term "IBN".

Recalling that any homomorphism $R^n \to R^n$ can be expressed by an $n \times m$ matrix via the natural bases on R^m and R^n, we can recast the definition (1.3) above in matrix terms. Thus, the ring R fails to have (right) IBN iff there exist natural numbers $n \neq m$ and matrices A, B over R of sizes $m \times n$ and $n \times m$ respectively, such that $AB = I_m$ and $BA = I_n$. One advantage of this statement is that it involves neither right nor left modules. In particular, we see that "right IBN" is synonymous with "left IBN". From now on, therefore, we can speak of the IBN property without specifying "right" or "left".

The zero ring is a rather dull example of a ring not satisfying IBN. C. J. Everett, Jr. was perhaps the first one to call attention to the following type of interesting examples.

[1] Hereafter, we shall abbreviate "finitely generated" by "f.g."

(1.4) Example. *Let V be a free right module of infinite rank over a ring $k \neq (0)$, and let $R = \mathrm{End}(V_k)$. Then, as right R-modules, $R^n \cong R^m$ for any natural numbers n, m.* For this, it suffices to show that $R \cong R^2$. Fix a k-isomorphism $\varepsilon : V \to V \oplus V$ and apply the functor $\mathrm{Hom}_k(V, -)$ to this isomorphism. We get an abelian group isomorphism

$$\lambda : R \to \mathrm{Hom}_k(V, V \oplus V) \cong R \oplus R.$$

We finish by showing that λ is a *right R-module homomorphism*. To see this, note that

$$\lambda(f) = (\pi_1 \circ \varepsilon \circ f,\ \pi_2 \circ \varepsilon \circ f) \quad (\forall f \in R),$$

where π_1, π_2 are the two projections of $V \oplus V$ onto V. For any $g \in R$, we have

$$\begin{aligned}\lambda(fg) &= (\pi_1 \circ \varepsilon \circ f \circ g,\ \pi_2 \circ \varepsilon \circ f \circ g)\\ &= (\pi_1 \circ \varepsilon \circ f,\ \pi_2 \circ \varepsilon \circ f)\, g\\ &= \lambda(f)\, g,\end{aligned}$$

as desired. An explicit basis $\{f_1, f_2\}$ on R_R can be constructed easily from this analysis. In fact, in the case when $V = e_1 k \oplus e_2 k \oplus \cdots$, we have essentially used the above method to construct such $\{f_1, f_2\}$ in *FC*-Exercise 3.14. In the notation of that exercise, we have also a pair $\{g_1, g_2\}$ with

$$g_1 f_1 = g_2 f_2 = 1, \quad g_1 f_2 = g_2 f_1 = 0, \quad \text{and} \quad f_1 g_1 + f_2 g_2 = 1.$$

This yields explicitly the matrix equations

$$(*) \qquad (f_1, f_2)\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = 1, \qquad \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}(f_1, f_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for checking the lack of IBN for R.

(1.5) Remark. Let $f : R \to S$ be a ring homomorphism. (This includes the assumption that $f(1) = 1$.) *If S has IBN, then R also has IBN.* In fact, if there exist matrix equations $AB = I_n$, $BA = I_m$ over R as in the paragraph following (1.2), with $n \neq m$, then we'll get similar equations over S by applying the homomorphism f, contradicting the IBN on S. Alternatively, we can also prove the desired result by applying the functor $- \otimes_R S$ to free right R-modules.

Now we are in a good position to name some classes of rings that have IBN.

(1.6) Examples.

(a) As we have mentioned before, *division rings have* IBN.

(b) *Local rings $(R, \mathfrak{m})$ have* IBN. This follows from (1.5) since we have a natural surjection from R onto the division ring $R/\mathfrak{m}$.

(c) *Nonzero commutative rings R have* IBN. In fact, if $\mathfrak{m}$ is any maximal ideal in R, then we have a natural surjection from R onto the field $R/\mathfrak{m}$.