LOGIC

with a Probability Semantics

Including Solutions to Some Philosophical Problems

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PREFACE

My interest in the connection of probability with logic was initiated by a reading of J. M. Keynes' *A Treatise on Probability*. This interest was furthered during a lengthy engagement with Boole's *Laws of Thought*, resulting in the writing of *Boole's Logic and Probability* (Hailperin 1976, 2nd ed. 1986). There is an extended historical presentation in *Hailperin* 1988 of matters pertaining to probability logic before its formalization as a logic. Subsequent publications on this topic included a book, *Sentential Probability Logic* (Hailperin 1996), containing a historical account but limited to the relation of probability with sentential logic. A number of publications then followed with further developments and applications, the present publication being an organized presentation of these further developments and includes an extension of probability logic to quantification language.

Published as separate papers over a period of time, a fair amount of editorial reorganization was needed to have them here all together as a unified subject. Nevertheless, being close enough to the original papers, I wish to express my thanks to the copyright owner-publishers for permission to so reproduce them. For papers in the Bibliography listing with dates 1988, 1997, 2006, 2007, 2008, I thank Taylor & Francis for this permission, and to Kluwer Academic Publishers for the one with date 2000. There is also some new material. Of particular note in that respect is the Main Theorem for probability logic (§3.2).

As with *Sentential Probability Logic*, Max Hailperin was a reliable resource for resolving TeX typesetting and formatting difficulties in producing the ms. He also supplied programs for producing the figures here in §§1.7, 3.4, 4.1 as well as obtaining, with use of the computer program *Mathematica*, the solution of the linear algebraic equation-inequation system in §1.7. For all this, and many helpful suggestions, I express my sincere appreciation and thanks.

Logic with a Probability Semantics

INTRODUCTION: AN OVERVIEW

1. This monograph is in large measure a continuation of *Sentential Probability Logic* (*Hailperin 1996*, hereafter to be cited as SPL). Its object is to extend the sentential probability logic there introduced, to the quantifier level and to present material making use of the extension. However the language with quantifiers we shall be using as a basis is not the customary well-known one but one that is in all respects ontologically neutral—in particular, there is no mention of individuals (arguments for predicates) or predicates. With its spare character, appealing only to the essential formal properties of quantifiers, one is able to focus exclusively on logical fundamentals. Application of this probability logic to languages that have sentences with individuals and predicates is not excluded—it just isn't required of a language that it must be of that form and, in particular as we shall see, the individuals-predicate structure of a sentence plays no role in our probability logic.

2. A large part of SPL is devoted to historical origins, to illustrative examples, to applications, and to related work of other authors. Rather than referring the reader to SPL to pick out from it those parts which are specifically needed for the present study, we thought it would be a convenience to have such material summarized here in an initial Chapter 1. At the same time this will afford us an opportunity to include some new material relating to sentential probability logic, as well as clarifying its exposition at a few places.

3. Probability logic as we view it is a form of logic which uses the same formal language as does verity logic but has a more general semantics. Our summarizing Chapter 1 opens with a description of the basic notions of verity sentential logic but so chosen as to foreshadow that for probability logic. Thus, instead of truth tables, verity functions are used to characterize how verity values (*true, false*) accrue to logical formulas when values are assigned to atomic sentence components. Analogously, our probability functions, a generalization of verity functions, serve to determine probability values for formulas. But now the assignment of probability values is not to atomic components but to constituents (e.g., if A_1 and A_2 are the two atomic sentences of a formula then its constituents are A_1A_2, $\neg A_1A_2$, $A_1\neg A_2$, $\neg A_1\neg A_2$). When properly specified (see §1.2 **II** in Chapter 1 below) such assignments are the probability models which determine a probability value for every sentence of the formal ($\neg$, $\wedge$, $\vee$)-language (supposing that $\neg$, $\wedge$, $\vee$ are its logical constants). Specific application of probability logic is made by selecting an appropriate probability model, just as for verity logic with an appropriate verity model.

It is to be emphasized that the probability logician's official business is not in the determination of what probability values are to be assigned to constituents so as to have a probability model, but rather that of the applied probabilist. She (or he) is the one who tosses the coin, draws balls from an urn, makes statistical analyses, or uses epistemic considerations, and chooses the probability model to be used. In probability logic probability values are assigned to sentences, not to (mathematical representations of) events as in mathematical probability theory.

4. The fundamental (inferential) notion of logical consequence for sentential probability logic (§1.2 **III** below) is a generalization of that for verity logic with its $V(\phi) = 1$, V a verity function, being replaced with $P(\phi) \in \alpha$, P being a probability function and α being a subset of the unit interval $[0, 1]$. Using this form as our basic semantic statement for probability logic—rather than an equality $P(\phi) = a$, a being a numerical value in $[0, 1]$—makes for a significant extension of the notion of logical consequence. For treating finite stochastic situations sentential probability logic is equally as capable as Kolmogorov probability spaces, where probability values are assigned to sets. (See §1.2 **IV** below)

5. Validity of (sentential) probability logical consequence being appropriately defined, of special interest is the result (stated in §1.3 below, carried over from SPL, §4.6) that when the subsets involved in a logical consequence statement are subintervals of $[0, 1]$ with explicitly given rational end points, then there is an effective procedure for determining whether or not the statement is a valid one.

6. Development of a conditional-probability logic extending probability logic entails first of all enlarging sentential logic with a new binary connective, the suppositional. It involves having a third semantic value in addition to 0 and 1, these two still retaining their meaning of *false* and *true*. The semantics for a 2-to-3 valued logic was described in SPL §0.3, and illustrated there in §0.4 with an early application by computer scientists to logic design of computer circuits. The *suppositional*, defined as a 2-to-3 valued binary connective, was formally introduced and compared with the two-valued truth-functional conditional. As a side observation, unconnected with our probability theme, it was noted that using the suppositional in place of the truth-functional conditional as an interpretation for the " If A, then B" of ordinary discourse can resolve some puzzling instances. An example of this was given in SPL, pp. 247–248 where, in briefly mentioning this topic, about which much has been written, we were duly mindful of Callimachus' epigram, as reported by Sextus Empiricus, "Even the crows on the roofs are cawing about the nature of the conditionals."[1]

However our central interest regarding the suppositional is not in the philosophical problem of explicating "implication" as used in ordinary discourse, but that of establishing a conditional probability logic, i.e., a logic which includes the notion of conditional probability. Here the suppositional connective plays a key role. Section 1.4 below summarizes suppositional logic and §1.5 the conditional-probability logic (of sentences without quantifiers) which is based on it.

7. An analysis of an old paradox—the so-called "statistical paradox"—brings out some differences between verity and probability logical consequence (§1.6, below). Not surprisingly, in the case of probability and particularly conditional-probability inferences, inconsistencies in probability

[1] *Kneale and Kneale 1962* p. 128.

value assignments to premise formulas are much harder to spot, especially if schematic letters for formulas or parameters rather than actual numbers, are used. Our discussion of this example brings this out.

8. The concluding section in Chapter 1 is a discussion of the topic of combining evidence—'evidence' being taken as the condition in a statement of conditional probability. The interest is in the relation between the conditional probability of two pieces of evidence first when they are taken separately and then when the two are combined. We give a real life instance of this, using evidence that arose in a sensational murder trial. While two separate pieces of evidence individually may not be particularly strong, but when combined the logical connections between them can produce, as shown by this example, a marked change.

9. Chapter 2 presents the verity quantifier logic serving as the basis for probability quantifier logic.[2] Its formal syntactic language, however, is not that of the usual first-order predicate kind, but for an ontologically neutral logic one in which the atomic sentences have, as in sentential logic, no prescribed logical or linguistic structure. Of course, in an application these sentences may be interpreted as being composed of the familiar predicates with individuals as arguments. The sentences of formal ON logic are distinguished not by subject predicate structure but by being tagged with a variety of subscripts (to be described). While this form of a quantifier logic was not originated for the purpose of viewing probability theory denumerably—as was Borel's purpose in a 1909 paper—it turned out to be aptly suitable. (See §§3.3, 3.7, below.)

As with sentential verity logic, so also with quantifier verity logic, quantifier formulas (closed ones) acquire truth values by means of verity functions and assignments of truth values to atomic sentences. The definition of a verity function now includes means for specifying truth values for quantified formulas. A model for ontologically neutral logic is, as with sentential logic, an assignment of a verity value (0 or 1) to each atomic sentence and, again, for any such model there is a unique verity function that provides a verity value for each closed formula (Theorem 2.11). Likewise, the definition of a

[2]Our exposition here includes developments improving the presentation in *Hailperin 1997* and in *2000*.

valid logical consequence is the same, namely, for every verity function, the premise formulas having verity value 1 implies that the conclusion formula also has value 1.

An axiomatic formulation for ontologically neutral quantifier logic is presented, outwardly no different from that for first-order predicate logic when expressed in terms of schematic letters for formulas (§§2.2, 2.3). What makes for a difference when actual ON formulas are involved is the semantics.

The suppositional connective (§1.4) is then adjoined to this quantifier logic, its semantics in this context specified, and some properties are derived (§2.4).

10. In Chapter 3 construction of a probability logic for quantifier languages proceeds along lines developed for sentential languages. First there is a definition of a probability function for the language, now including in the list of its properties an additional one for sentences with quantifiers (P_4 in §3.1). A fundamental, though somewhat difficult theorem to prove, establishes that a probability model for such a language, appropriately defined, can be uniquely extended so as to be a probability function on the language (§3.2).

A special historically based section with heading "Borel's denumerable probability" (§3.3) makes use of material from the first serious consideration of "infinite" cases in probability theory. Our probability logic for quantifier language is ideally suited for this "infinite" aspect of Borel's probability theory as the central properties and results are readily incorporated in our basic definitions, or in readily established theorems.

There is a section (§3.5) that compares the Kolmogorov spaces approach to probability theory with the logical representation here to be presented. A discussion of Bernoulli's law of large numbers is used as an example for comparison.

Since the new quantifier logic which was introduced can have not only verity but also probability as a semantic value, a general discussion of the fundamental notion of logical consequence seemed needed. This is the content of §3.6.

11. A section (§3.7) is devoted to a comparison of our quantifier prob-

ability logic and the "denumerable probability" of *Borel 1909* and to a response to the critique of Borel's ideas contained in *Barone & Novikoff 1978*.

12. Our last extension of logical language adds the suppositional to quantifier language, and defines a suitable probability function for the language. In the resulting extended probability logic we offer a resolution of the "paradox of confirmation"— its origins dating back to 1945—on which many philosophical papers have been written.

13. In concluding this Introduction we acknowledge that there are probabilists who find the Kolmogorov axiomatic set-measure-theoretic basis for probability theory so successful that it is hard for them to envision room for any other. Here, for example, is an expression of the dubiousness of ventures that have probability on sentences (*Łoś 1955*, p. 135).:

> The disadvantages of practising the calculus of probability on sentences, which discourage mathematicians from proceeding in this direction, are as follows:
>
> 1° Fields of sentences are always denumerable what implies that a Boolean algebra composed of sentences is at most denumerable. This renders it impossible to conduct the probabilistic investigation with regard to problems which require a non-denumerable field of events; in particular such a theory makes it impossible to reconstruct Lebesgue's measure on an interval (see §1).
>
> 2° An algebra formed of sentences cannot be a σ-algebra, for it is easy to show that no denumerable Boolean algebra (i.e. of the power $\aleph_0$ exactly) is a σ-algebra.
>
> 3° The fact that events are sentences has never been properly utilized in the theory of probability built on sentences (with the exception of philosophical speculations of dubious value).

With regard to Łoś's 1°, just because a theoretical approach is capable of handling problems of varying degrees of complexity doesn't mean that it has to. Thus continuum mathematics, making use of the nondenumerable real number system doesn't need to be employed for all problems using numbers—e.g., those involving just the rationals, or just the integers. Not just aesthetics but also understanding is compromised by employing stronger assumptions and methods than what are needed.

Moreover, what meaning is there to a field of a non-denumerably many *events*? One can (and many do) conceive of a *mathematical abstraction* which is a set of non-denumerably many real numbers, but of that many *events*? As will be noted in a footnote in §3.5 below, Kolmogorov's response to an objection of this nature was to declare it "unobjectionable from an empirical standpoint" since the uses of these "ideal elements", as how he refers to them, leads to the determination of the probability of actual events.

As for 2°, while the notion of σ-additivity (of probability on sets) does not directly apply to probability logic in quantifier sentences, we do have a close substitute namely 'countable additivity'. Theorem 3.43 below shows that the disjunction of any specified denumerable sequence of mutually exclusive sentences, each with a probability value, has a determined probability value.

Coming to 3°, the reference decrying those who utilize the "fact that events are sentences" does not apply to us. In our probability logic sentences may refer to, or denote, events but are not identified with them, anymore than in verity logic are sentences identified with, i.e. as being, the facts or events described.

CHAPTER 1

SENTENTIAL PROBABILITY LOGIC

§1.1. Verity logic for $\mathcal{S}(\neg, \wedge, \vee)$

We begin with a brief presentation from a semantic viewpoint of the sentential portion of verity (true, false) logic. The emphasis is on the fundamental notion of logical consequence. After stating the definition of this notion for verity logic in customary form it is then slightly modified in order to bring out that the definition of logical consequence for probability logic to be introduced in §1.2 is, as a generalization, naturally related to it. A reason why the comparison is being made on a semantic (logical consequence) basis, rather than a formal syntactic (linguistic structure) one, is given below in item **V**, end of §1.2.

The symbols of our formal sentential language $\mathcal{S}$ are standard: a potential infinite sequence of atomic sentences $A_1, \ldots, A_n$ which can be of any length, together with logical connectives $\neg$ (not), $\wedge$ (and), $\vee$ (or), in terms of which the sentences of $\mathcal{S}$ are compounded.[1] As for its semantics there are the two verity values 0 and 1. These are assumed to have the arithmetical properties of the numbers 0 and 1 so as to conveniently express the logico-semantic properties of *false* and *true*. Not only for this reason are these two numbers useful but they also will enter as semantic values for probability logic, along with the numbers in-between so that, as we shall see, sentential probability logic is a natural extension of sentential verity logic. To define the semantic properties of the connectives for verity logic

[1] As is well known, other choices of logical connectives can be made. These shown here are somewhat more convenient for depicting the evolvement of probability logic from verity logic.

we make use of the notion of a verity function. A function $V\colon \mathcal{S} \to \{0, 1\}$, from the sentences of $\mathcal{S}$ to the set of verity values $\{0, 1\}$, is a *verity function* on S if it has for any sentences ϕ and ψ of $\mathcal{S}$, the following properties:

$$\begin{aligned} V(\neg\phi) &= 1 - V(\phi) \\ V(\phi \wedge \psi) &= \min\{V(\phi), V(\psi)\} \\ V(\phi \vee \psi) &= \max\{V(\phi), V(\psi)\}. \end{aligned}$$

It is then easy to show that with respect to any given verity function the connectives have their usual properties as defined by truth tables, and that for a given V once values are assigned to the atomic sentences of $\mathcal{S}$, then every sentence of $\mathcal{S}$ has a uniquely defined V value. An assignment of verity values to the atomic sentences of $\mathcal{S}$ is a *verity model.* If M is a verity model we write 'V_M' for the uniquely determined associated verity function which assigns a verity value to each formula of $\mathcal{S}$. Conversely, for any verity function V defined on $\mathcal{S}$ there is a uniquely determined model M, namely that which assigns to an atomic sentence the value that the V does.[2]

For $\psi, \phi_1, \ldots, \phi_m$ that are sentences of $\mathcal{S}$, we define ψ to be a *verity logical consequence* of $\phi_1, \ldots, \phi_m$ if the following holds:

$$\begin{aligned} &\text{for all models } M, \\ &\quad \text{if } V_M(\phi_1) = 1, \ldots, V_M(\phi_m) = 1, \text{ then } V_M(\psi) = 1. \end{aligned} \tag{1}$$

The customary notation for this property is '$\phi_1, \ldots, \phi_m \models \psi$' or simply '$\models \psi$' if there are no premises (i.e., when ψ is truth-functionally valid). Note that the symbol '$\models$' carries the entire semantic meaning that is expressed more fully by (1). Since to each verity model there uniquely corresponds a verity function we can replace 'for all models M' by 'for all verity functions V', drop the subscript 'M' from 'V_M' and, understanding the initial universal quantifier 'for all verity functions V', reduce (1) to the simple

$$\text{if } V(\phi_1) = 1, \ldots, V(\phi_m) = 1, \text{ then } V(\psi) = 1 \tag{2}$$

[2]Note that $\mathcal{S}$, M and V_M are defined in terms of $A_1, \ldots, A_n$ for any $n \geq 1$, though there is no indication of this on the symbols.

or the simpler

$$\phi_1, \ldots, \phi_m \ \vDash \ \psi, \tag{3}$$

widely used to depict this notion. This definition of verity logical consequence will be of interest by way of comparison when we come to state the one for probability logical consequence where the notion of probability function replaces that of verity function.

The notion of logical consequence to be introduced in the next section, where 'probability' replaces 'verity', bears little resemblance to (2). But the existence of a relationship can be brought out by generalizing components of the form '$V(\phi) = 1$' in (2) to '$V(\phi) \in \alpha$', where α can be any non-empty subset of $\{0, 1\}$. Making such a change in the premises results in nothing new since $V(\phi) \in \{1\}$ is equivalent to $V(\phi) = 1$, $V(\phi) \in \{0\}$ is equivalent to $V(\neg\phi) = 1$ and any $V(\phi) \in \{0,\ 1\}$, having vacuous content, i.e., being true for any ϕ, can be deleted. Though if $V(\psi) \in \{0,\ 1\}$ were to occur in the conclusion then some information is conveyed if it were the strongest conclusion that can be obtained, namely that the information presented in the premises are insufficient to determine a verity value for ψ. While such a generalization to sets of truth-values as just described has no significance in the case of verity logical consequence, we shall later see that for the case of probability logical consequence, a marked enlargement of the range of application of the notion of consequence occurs when using subsets of probability values rather than just a single value.

From this sketch of verity logical consequence we now turn to that for our probability logical consequence which, though a natural generalization of the one for verity logic, is more complicated.

§1.2. Probability logic for $\mathcal{S}(\neg, \wedge, \vee)$

I Probability Functions. Although '$\vDash \phi$' is customary for expressing validity of ϕ in formal verity logic, instead of it we shall now use '$\vDash_{tf}\ \phi$' to express that ϕ is truth-functionally valid. This frees the bare '$\vDash$' for us to use with probability logic, obviating then the need for two distinguishing marks on '$\models$' since, as we shall see, both kinds of validity can be occurring though that for sentences being quite subsidiary.

For sentential probability logic the formal syntax language is $\mathcal{S}$, the same as that for verity logic. In $\mathcal{S}$ we will make use of the truth-functional conditional '$\rightarrow$' and equivalence '$\leftrightarrow$' both defined as usual in terms of $\neg$, $\wedge$, $\vee$. In place of verity functions we now have probability functions defined as follows.

Corresponding to the function V mapping sentences of $\mathcal{S}$ onto the two element set $\{0, 1\}$, here we have a function $P\colon \mathcal{S} \rightarrow [0,\, 1]$, from sentences of $\mathcal{S}$ to the reals of the unit closed interval $[0, 1]$, defined to be a *probability function on* $\mathcal{S}$ if it has the following properties:

For any ϕ and ψ of $\mathcal{S}$,

P1. If $\models_{tf} \phi$, then $P(\phi) = 1$.

P2. If $\models_{tf} \phi \rightarrow \psi$, then $P(\phi) \leq P(\psi)$.

P3. If $\models_{tf} \phi \rightarrow \neg\psi$, then $P(\phi \vee \psi) = P(\phi) + P(\psi)$.

It then readily follows that if $\models_{tf} \phi \leftrightarrow \psi$, then $P(\phi) = P(\psi)$ and, more generally, where ϕ' comes from ϕ by replacing an occurrence of ρ by σ, that

$$\text{if } \models_{tf} \rho \leftrightarrow \sigma, \text{ then } P(\phi) = P(\phi').$$

It is readily seen (SPL, p. 190, Theorem 4.22)[3] that a probability function whose range is the two-element set $\{0, 1\}$ is a verity function, and that every verity function is a probability function. Clearly the properties of probability functions, having ranges that are subsets of [0,1], are much more extensive than those of verity functions. Here is a list of some easily derivable elementary ones:

(a) $P(\phi \vee \neg\phi) = 1$

(b) $P(\phi) + P(\neg\phi) = 1$

(c) $P(\phi) = P(\phi\psi) + P(\phi\overline{\psi}) \qquad (\overline{\psi} := \neg\psi,\ \phi\psi := \phi \wedge \psi)$

(d) $P(\phi \vee \psi) = P(\phi) + P(\psi) - P(\phi\psi)$.

[3]Recall that 'SPL' stands for *Sentential Probability Logic*, i.e., *Hailperin* 1996.

There are alternative ways of expressing the defining properties of a probability function on $\mathcal{S}$: sacrificing the nice parallelism in the hypotheses of P1–P3 one can replace P2 by (b) and P3 by (d) of the above list and have an equivalent definition of a probability function on $\mathcal{S}$. (In the parenthetical remark accompanying (c) the symbol ':=' is an abbreviation for 'is defined as'.)

II Probability Models. The definition of logical consequence for probability logic makes use of probability models. However, unlike verity models which are assignments of verity values to the atomic sentences of $\mathcal{S}$, in the case of *probability models* the probability values are assigned to *constituents of* $\mathcal{S}$ ("constituents" is what Boole, who introduced them into logic, called them). A constituent of $\mathcal{S}$ is a logical product of m atomic sentences, e.g. $B_1, \ldots, B_m$ in which none, some, or all are negated so that for m atomic sentences there are 2^m of such products. A compact notation for such a product is $K_{b_1 \cdots b_m}$ where b_i $(i = 1, \ldots, m)$ is either 1 or 0, it is 1 if B_i is unnegated and 0 if it is negated. In determining validity for verity logic it is immaterial whether verity values are assigned to the $B_1, \ldots, B_m$ or to the 2^m constituents on $B_1, \ldots, B_m$ since an assignment of verity values to the m $B_1, \ldots, B_m$ determines that of a unique constituent, and conversely; if B_i is assigned the value 1 then B_i is to appear as a conjunct in the constituent, but if B_i is assigned the value 0, then $\overline{B}_i$ is to appear in the conjunct. This is not the case with probability values where it only works if the components of the constituents are probabilistically independent.[4]

Let $\mathcal{S}(K)$ denote the set of constituents of $\mathcal{S}$. A *probability model* is a function $M\colon \mathcal{S}(K) \to [0,\,1]$ which for each i $(i = 1, 2, \ldots, m)$ assigns a value $k_{b_1 \cdots b_i}$ in $[0, 1]$ to $K_{b_1 \cdots b_i}$ such that

$$\begin{aligned} &\text{(i)} && k_{b_1 \ldots b_i} \geq 0 \\ &\text{(ii)} && k_1 + k_0 = 1 \\ &\text{(iii)} && k_{b_1 \ldots b_i 1} + k_{b_1 \ldots b_i 0} = k_{b_1 \ldots b_i}. \end{aligned}$$

[4]Boole's General Method for solving "any" probability problem did assume (no justification was given) that any event could be analyzed as a logical compound of "simple" *independent* events, for which case it would be that their probability values do determine that of constituents on these events. (See SPL, §2.5, pp. 114-115.)

It then follows that for any m

$$\sum k_{b_1 \ldots b_m} = 1,$$

the sum being taken over all 2^m terms $k_{b_1 \ldots b_m}$ obtained as each b_i, $i = 1, \ldots, m$, takes on the value 0 or 1.[5]

Note that the values assigned to the $K_{b_1 \cdots b_m}$ may be, but need not necessarily be, determined by chance experiments or trials. Probability logic is accomodative to other kinds of interpretation for the notion of "probability" besides that of physical chance, for example, as an estimate of certainty[6] when there is inadequate knowledge.

Analogous to the extension of verity models for $\mathcal{S}$ to verity functions on $\mathcal{S}$ we have the result that an assignment of probability values to a specified subset of $\mathcal{S}$, i.e., those that are constituents, establishes one which satisfies P1–P3 for all formulas of the subsets of $\mathcal{S}$ (SPL, Theorem 4.41, taking its $\mathcal{S}$ to be the one here under discussion):

Any probability model M for $\mathcal{S}$ can be uniquely extended to be a probability function P_M on $\mathcal{S}$.

And, in the other direction, that one has

Any probability function P determines a probability model

is clear since properties (i) and (ii) of the definition of a probability model obviously hold and (iii) does by item (c) above with ϕ and ψ appropriately specified.

III PROBABILITY VALIDITY. The definition of logical consequence for sentential probability logic which we now state, is a generalization of that for verity logic (that of §1.1) in which probability models replace verity models, and subsets of $[0, 1]$ replace subsets of $\{0, 1\}$.

Let $\phi_1, \ldots, \phi_m, \psi$ be sentences of $\mathcal{S}$. We say ψ is *a probability logical consequence of* the m sentences $\phi_1, \ldots, \phi_m$ (with respect to non-empty subsets

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[5] As a help in keeping matters clear note that a b_i is a numeral 0 or 1, that $b_1 \cdots b_i$ is a numeral in binary notation, that $k_{b_1 \cdots b_i}$ a real number in [0,1], and $K_{b_1 \cdots b_i}$ a sentence in $\mathcal{S}(K)$.

[6] An idea going back at least to J. Bernoulli. See e.g., SPL §1.2.