One, Two, Three

Absolutely Elementary Mathematics

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Author of *A Tour of the Calculus*

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ABSOLUTELY ELEMENTARY MATHEMATICS

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For Neal Kozodoy

We read to find out what we already know.

—V. S. Naipaul

Contents

This is a little book about absolutely elementary mathematics (**AEM**); and so a book abo the natural numbers, zero, the negative numbers, and the fractions. It is neither a textbook, treatise, nor a trot. I should like to think that this book acts as an anchor to my other boo about mathematics.

Mathematicians have always imagined that mathematics is rather like a city, one who skyline is dominated by three great towers, the state ministries of a powerful intellectu culture—our own, as it happens. They are, these great buildings, devoted to Geometr Analysis, and Algebra: the study of space, the study of time, and the study of symbols a structures.

Imposing as Babylonian ziggurats, these buildings convey a sacred air.

The common ground on which they rest is sacred too, made sacred by the scuffle of huma feet.

This is the domain of absolutely elementary mathematics.

Many parts of mathematics glitter alluringly. They are exotic. Elementary mathematics, the other hand, evokes the very stuff of life: paying bills, marking birthdays, dividing deb cutting bread, and measuring distances. It is earthy. Were textbooks to disappear tomorro and with them the treasures that they contain, it would take centuries to rediscover t calculus, but only days to recover our debts, and with our debts, the numbers that expre them.

Elementary mathematics as it is often taught and sometimes used requires an immersi into messiness. Patience is demanded, pleasure deferred. Decimal points seem to wand negative numbers become positive, and fractions stand suddenly on their heads.

And what *is* three-fourths divided by seven-eights?

The electronic calculator has allowed almost everyone to treat questions such as this wi an insouciant indifference. Quick, accurate, and cheap, it does better what one hundred yea ago men and women struggled to do well. The sense that in elementary mathematics thin are familiar—half remembered, even if half forgotten—is comforting, and so are t calculator and the computer, faithful almost to a fault, but the imperatives of memory a technology do prompt an obvious question: why bother to learn what we already know or least thought we knew?

The question embodies a confusion. The techniques of elementary mathematics are o thing, but their *explanations* are quite another. Everyone can add two simple natural numbe together—two and two, for instance. It is much harder to say what addition means and w it is justified. Mathematics explains the meaning and provides the justification. The theori that result demand the same combination of art and sophistication that is characteristic of a great intellectual endeavor.

It could so easily have been otherwise. Elementary mathematics, although pressing in i urgency, might have refused to cohere in its theory, so that, when laid out, it resembled map in which roads diverged for no good reason or ended in a hopeless jumble. But t theory by which elementary mathematics is explained and its techniques are justified intellectually coherent. It is powerful. It makes sense. It is never counter-intuitive. And so is appropriate to its subject. If when it comes to the simplest of mathematical operations-addition again—there remains something that we do not understand, that is only becaus there is nothing in nature (or in life) that we understand as completely as we might wish.

Nonetheless, the theory that results is radical. Do not doubt it. The staples of childhoo education are gone in the night. One idea is left, and so one idea predominates: *that t calculations and concepts of absolutely elementary mathematics are controlled by the single act counting by one.* There is in this analysis an economy of effect, and a reduction of experienc to its essentials, as dramatic as anything found in the physical sciences.

Until the end of the nineteenth century, this was not understood. A century later, it is st not *widely* understood. School instruction is of little help. "Please forget what you hav learned in school," the German mathematician Edmund Landau wrote in his book *Foundatio of Analysis;* "you haven't learned it."

From time to time, I am going to ask that readers do some forgetting all their own.

A secret must now be imparted. It is one familiar to anyone writing about (or teachin mathematics: no one very much likes the subject. It is best to say this at once. Like ches mathematics has the power to command obsession but not often affection.

Why should this be—the distaste for mathematics, I mean?

There are two obvious reasons. Mathematics confronts the beginner with an aura strangeness, one roughly in proportion to its use of arcane symbols. There is something abo mathematical symbolism, a kind of peevishness, that while it demands patience, seems hard to promise pleasure.

Why bother?

If the symbolic apparatus of mathematics is one impediment to its appreciation, th arguments that it makes are another. Mathematics is a matter of proof, or it is nothing. B certainty does not come cheap. There is often a remarkable level of detail in even a simp mathematical argument, and, what is worse, a maddening difference between th complicated structure of a proof and the simple and obvious thing it is intended demonstrate. There is no natural number standing between zero and one. Who would dou it? Yet it must be shown, and shown step by step. Difficult ideas are required.

Why bother?

A tricky bargain is inevitably involved. In mathematics, something must be invested befo anything is gained, and what is gained is never quite so palpable as what has been invested. is a bargain that many men and women reject.

Why bother indeed?

The question is not ignominious. It merits an answer.

In the case of many parts of mathematics, answers are obvious. Geometry is the study space, the mysterious stuff between points. To be indifferent to geometry is to be indifferen to the physical world. This is one reason that high-school students have traditionally accepte Euclid with the grudging sense that they were being forced to learn something that the needed to know.

And algebra? The repugnance (in high school) that this subject evokes has always bee balanced by the sense that its symbols have a magical power to control the flux and fleen things. Farmers and fertilizers were the staple of ancient textbooks. But energy and ma figure in those that are modern. Einstein required *only* high-school algebra in creating h theory of special relativity, but he required high-school *algebra,* and he would have been lo without it.

Mathematical analysis came to the mature attention of European mathematicians in th form of the calculus. They understood almost at once that they had been vouchsafed the fir and in some respects the greatest of scientific theories. To wonder at the importance o analysis, or to scoff at its claims, is to ignore the richest and most intensely developed bod of knowledge acquired by the human race.

Yes, yes. This is all very uplifting, but absolutely elementary mathematics? Not very lon ago, the French mathematician Alain Connes invented the term *archaic mathematics* t describe the place where ideas are primeval and where they have not yet separate themselves into disciplines. It is an elegant phrase, an apt description. And it indicates ju why elementary mathematics, when seen properly, has the grandeur of what is absolute. It fundamental, and so, like language, an instinctive gesture of the human race.

A *theory* of absolutely elementary mathematics is an account in modern terms of somethin deep in the imagination; its development over the centuries represents an extraordinar exercise in self-consciousness.

This is what justifies the bothering, the sense that, by seeing an old, familiar place throug the mathematician's eyes, we can gain the power to see it for the first time.

This is no little thing.

—Paris, 2010

1

One sheep, two sheep, three. Wool to follow …

NUMBERS

The natural numbers 1, 2, 3, …, play a twofold role in our ordinary affairs. Without then there could be no counting, and so no answer to the question *How many?* A man who unable to tell whether he is looking at one sheep or two of them cannot *identify* sheep. He left staring at so much wool on the hoof. It is the natural numbers that offer him relief fro sheeplessness. "The creation of numbers," Thierry of Chartres remarked in the twelf century, "was the creation of *things.*"

As counting endows things with their identity, so it imposes on them their difference. Thre sheep make for three *things.* The natural numbers are the expression in nature of division ar distinctness. Between the number one and the number two there is, after all, nothin whatsoever, and nothing between things that are distinct either, however much alike the might be in various respects. The discreteness of the natural numbers is as absolute as the or enforced by the surface of our skin, which permits contact but not, alas, commingling.

There are certainly substances in the world that cannot be counted—mud, for example. Th word 'mud' seems indifferently to designate mud wherever it is and however it may b found. But so strong is the intellectual impulse to subordinate experience to counting th ordinary English provides the tools by which even mud can be made numerate—a *spot* mud, or a *dab,* or a *pile,* whereupon there is *one* spot, *two* dabs, or *three* piles. The same *on two,* and *three* used in counting sheep are also used in sorting them. It is the natural numbe that make it possible for some leather-faced Spanish sheepherder, his concave cheeks pursin around two gold teeth, to put his sheep and thus his life in order.

The *first* is mine, *hombre,* as these sheepherders say, the *second* yours, and the *third* his.

A SCRIBAL ART

The Sumerians drilled their children in **AEM** more than five thousand years ago, when th desert sun was new and nothing was yet old. Sumerian children were taught the basics; the teachers had grasped the essentials. They did not find it easy. Sumerian scribes studied fo years beyond childhood in order to hen-scratch clay tablets with tax records, business claim legal codes, real-estate transactions. They left behind a sense of their mathematical intimac the first in history.

An inadequate sense of their calling was not among their afflictions. "The scribal art," or wrote, "is the father of masters."

Scribes *alone,* he added, could "write [inscribe] a stele; draw a field, settle accounts."

There is a gap in the text, a break in its flow.

And then a phrase isolated on both sides, one suggestive of the scribe's intellectu grandeur: … *the palace* …

At the end of the third millennium b.c., the Sumerian empire ran streaming into the dese sands, defeated at last by time. Carried by the wind, I suppose, or some other current warm thought, the scribal sense of intimacy with **AEM** was acquired by Chinese mandarin intoxicated by their new power over pictograms, and acquired again by the Babylonians, s that it appears throughout the ancient world.

Different societies used **AEM** in their own ways and for their own ends. Every socie missed something, and no society, not even our own, knew or knows it all.

A MAN APART

Leopold Kronecker was born in 1823, his birthplace, the small city of Liegnitz, in Ea Prussia. Having quaked to the sound of Russian tanks in late 1944, Liegnitz is now known a Legnica. It is a part of Poland. East Prussia has vanished.

Kronecker's face is not easy to read on a photograph. Harsh lighting and extended exposu have darkened and deepened every facial line. The stern creases suggest an unacknowledge blood tie between Leopold Kronecker and General William Tecumseh Sherman. In both me the forehead is high, and the hair cut short, almost *en brosse;* the eyes are deeply recessed an melancholy. In all this, Kronecker, at least, is completely Prussian and austere, but his no has undertaken a racially unmistakable life of its own, hooking proudly at the bridge, an then curving toward its fluted tip.

I mention this not in order to make fun of another man's nose—I have a give-away nose o my own, after all—but to convey something of Kronecker's capacity to stand apart fro other mathematicians while standing among them. Kronecker was that rare character in th history of thought, a *mathematical* skeptic, unwilling to countenance ideas that he could no completely grasp, and very quick to conclude that he could not grasp most ideas completel If Kronecker the Glum was notable for saying *no*—*no* to the negative numbers, *no* to the re numbers, *no* to sets—he was notable for saying *yes* to the natural numbers, a great lif affirming *yes* to these ancient objects of thought and experience, a *yes* spilling over t encompass any mathematical construction that returned to the natural numbers in a fini series of steps.

Kronecker, the man of a thousand *no*'s, and Kronecker, the man of a single *yes,* we combined in a singular personality: suave, supple, self-satisfied.

While still in his twenties, Leopold Kronecker pursued a career in business as the manag of his uncle's estates in East Prussia. He had a remarkable gift for practical affairs, and ov the course of eight years, he made himself a wealthy man. Thereafter, he bought a splend Berlin mansion, and after marrying his uncle's daughter, Fanny Prausnitzer, made it a cent of culture and refinement.

Wealth made Kronecker indifferent to the great game of mathematical chairs in which th leading mathematicians of Europe stood looking eagerly at a small number of seat cushio still glowing with the warmth of some departed professor's buttocks. When the mus stopped, they scrambled unceremoniously for the vacant seat. Inevitably, most we disappointed. Mathematicians of genius, such as Georg Cantor, spent years waiting for a ca from Berlin and were bitterly vexed when it did not come.

Herr Kronecker expressed no very great interest in becoming Herr Professor. He did n need to scramble for his seat. Or for his supper. What he lacked was the right to lecture the University of Berlin. This he wished very much to have. Devoted to topics in numb theory, elliptic functions, and algebra, his papers were in every way remarkable without any way being revolutionary. When he was elected to the Berlin Academy in 1861, he gaine the right to lecture at the university.

Having declined to mount the greasy pole, he found himself at its very top. Once there, h determined to persecute those with whom he disagreed. It was an activity he carried out wi never-flagging zeal.

IN EVERY HUMAN MIND

At the very beginning of human history, a Neolithic hunter chipped a number of slash mark or tallies onto his ax handle. Was he recording bison killed? I do not know. I like to thin that as *my* ancestor, he had a contemplative nature, and regarded the numbers as things themselves, leaving those bloated bison to his rivals.

If the natural numbers appear at the very beginning of human history, they also appea spontaneously in every human mind. Otherwise, arithmetic could not be taugh Anthropologists are often amazed by the radically incommensurable way in which differen societies organize the most basic facts of experience. Seeing this is said to be one of th pleasures of travel. Nonetheless, our own *one, two, three,* the Latin *unus, duo, tres,* and th Akkadian *dis, min, es,* designate precisely the same numbers. If goat eyes are a delicacy Khartoum but not in New York, it is nonetheless true that three goat eyes is one more tha two in both cities.

Because they are universal, the natural numbers very rarely are the cause of introspectio We take them for granted. We would be lost without them.

There they are.

What they are is another matter entirely.

The English logician and philosopher Bertrand Russell was a passionate opponent of th First World War, and he took the occasion of his confinement as a conscientious objector organize his thoughts about the nature of the numbers. It may be imagined that Russell wa writing under conditions of personal austerity, but in his *Autobiography* he indicates tha except for his freedom, he was provided every comfort by his jailers.

The book that Russell wrote in prison, his *Introduction to Mathematical Philosophy,* is a wor of logical analysis. It has had a very great influence among mathematicians and philosopher because it offers an account of the natural numbers in terms of something *other* than th natural numbers. Such an account is needed, Russell believed, because the numbers a "elusive" in their nature, and though they make their influence felt in the most ordinary activities—counting sheep, after all—what they are doing is far easier to determine than ho they are doing it.

For one thing, the numbers are not physical objects. They are not objects at all. Thre sheep are in the pasture. There are not, in addition to the sheep, three numbers loiterin around and munching grass.

Nor, however, are the numbers properties of physical objects. Three sheep are three number, just as they are white in color. This is a step in the right direction. But to argue th being three is just like being white invites the question of just what property makes thre sheep three. We know what makes them white: it is their color. To say that what makes the three is their *number* does not seem a step forward. If we knew what the numbers were, w would know what three of them add up to.

To the question of just what makes three sheep three, Russell argued that those three shee were similar to other sets of three things—triplets, troikas, or trios. This is obviously s Three sheep and three sheepherders *are* alike. There are three of them. Russell next argue that being alike in being three could be defined with no appeal to the number three. This the crucial step. Three sheep and three sheepherders are alike if each sheepherder can b matched with one and only one sheep, and vice versa. Numbers are not required. No shee lacks a herder and no herder a sheep.

This is ingenious, but it is also disappointing. The number three was destined to disappea in favor of similar sets, but what, after all, makes a set of three sheep a set of *three* sheep ar not four? Four sheepherders and four sheep may also be put into correspondence so th nothing is left over and no sheep or sheepherder left out. The obvious answer is that a set four sheep is larger than a set of three sheep.

It is, in fact, larger by precisely *one* sheep.

WIVES, GOATS, NUMBERS

The natural numbers begin at one; they increase by one; and they go on forever. Th knowledge that this is so is an inheritance of the race. Anthropologists, it is true, report th certain tribes lack a complete sense of the numbers. Men count in the fashion of *one, tw many,* and they refer to any number past two by observing glumly that it is many. *One chie two goats, many wives,* as a great chief might say.

I am skeptical of such reports, because I feel quite certain that stealing one of the chief wives would prompt the chief to observe that he has *one less* than many wives. If he capable of determining that he has *one less* wife than he might need, he is equally capable determining that he has *one more* wife than he might want. Pressed thus by the exigencies tribal life, he could count up by enumerating his ensuing discontent: *many wives, one mo than many wives, one more than one more than many wives,* and so on toward frank domest nightmare.

Going in the other direction, he could count down until he reached bedrock in the numbe one, whereupon he could compare the number of his squabbling wives with the number of h chiefs—one in both cases. It is a laborious system, to be sure, but brain workers are ofte indifferent to practical concerns.

GOD'S WORK

If it is difficult to say what the numbers are, it is difficult again to say how they are used. Th most familiar way of counting a small number of sheep is to match those sheep to the tips one's fingers as they uncoil from a fist; it is what we all do when we are minded to cou sheep. But as an explanation of counting sheep, it suffers the drawback that counting one fingers, however familiar, is no easier to explain than counting one's sheep. Shall we expla counting three sheep by an appeal to counting sheep one at a time, with counting by or explained in terms of a physical act—moving those sheep one by one from pasture paddock, for example? It has an appeal, this sort of explanation. Something is being don and when it has been done, someone has done something. But obviously, if we wish t understand what it means to count three sheep, it will hardly improve our grasp of the matte to be told that we must first count sheep one by one *three* times. What holds for countin holds for ordering too, as when a shepherd's ever-useful fingers are used to explain the fa that the first sheep into the paddock comes *before* the second, and the second *before* the thir Fingers are vigorously extended: the first, the second, and the third. Yet, if sheep a following in a certain order, it is hardly to the point to appeal to the *same* order amon fingers as among flocks. If the order is not the same between fingers and flocks, of what us are the fingers? If it is the same, of what use the analogy?

At a certain point—*now,* perhaps—it becomes reasonable to suppose that neither th numbers nor the operations they make possible permit an analysis in which they disappear favor of something more fundamental. It is the numbers that are fundamental. They may b better understood; they may be better described; but they cannot be *bettered.*

The natural numbers, Leopold Kronecker remarked, are a gift from God. Everything else the work of man. This is a radical position in thought—an admission, on the one hand, th the natural numbers cannot be explained, and a suggestion, on the other, that th mathematician's proper work must be to accept this strange gift and from it derive all others

It is comforting to realize that in **AEM** we are doing God's work.

2

Henry had six wives but ‘Henry’ has five letters. There is a distinction between numbers and their names. Without the distinction, it is difficult to understand how numbers are named and impossible to understand the ancient and civilized art of positional notation.

POSITIONS IN POWER

The distinction is not an easy one to grasp, even for mathematicians, and it is a difficu distinction to maintain, even for me. What was it that Horace said? Even Homer nods. I ‘Henry’ has five letters, it is the name that counts; in Henry had six wives, the man. Logiciar and philosophers use single quotation marks to specify Henry’s name, but in this boo boldface is at work to the same end—**Henry** *versus* Henry, and **1** *versus* 1.

The distinction between names and numbers often slips, and with it the mathematiciar Thus in a very interesting book entitled *What Is Mathematics, Really?,* Reuben Hersh defin the relationship of equality between numbers by an appeal to the formulas in which they a named. Two *numbers* are equal, Hersh writes, “if in any *formula,* [one] may be replaced b [the other] and vice versa.”[2]

This could not be true; it is not so. Formulas are symbolic forms: marks on paper, soun on the moving air, or even lines within a computer program. The number four cannot repla anything in any formula. Only symbols can replace symbols.

On the other hand, the number four was equal to itself long before formulas were existence; and equal to itself, for that matter, before the earth cooled, or the solar syste formed, or the universe erupted into being out of nothingness.

Numbers owe their identity to no symbolic contrivance. They are what they are. They hav always been what they always were. They are not destined to change. But the numerals (ar so the formulas) name; they denote; they designate; they are a part of the apparatus wi which we make up a world of symbols in order to represent a world of things. How symbo designate numbers is mysterious, because we do not understand how names designate thing Speaking in *The Book of the Thousand and One Nights,* a Prince offers as good an explanatic as any: There is no letter in any language, he remarks, “which is not governed by a spirit, ray, or an emanation of the virtue of Allāh.”

A MAN FOR ALL SYMBOLS

The notation used to name the natural numbers is Hindu-Arabic, and it seems to have gaine currency among mathematicians during the early part of the ninth century.

The man most associated with the transmission of the Hindu-Arabic numerals to the west Abu Ja’far Muhammad ibn Musa al-Khwarizmi. Born at some time during the latter part the eighth century, and dying at some time during the middle of the ninth, al-Khwarizmi w one of those superbly accomplished scholars whose life remains much improved by a incomplete biographical record. Although he wrote in Arabic, he may have been Persian origin; some authorities suggest a Zoroastrian affiliation. A contemporary stamp with h likeness depicts a turbaned character with a long, severe face, the nose aquiline in the spir of great curved mathematical noses, and a curly beard descending in tight ringlets. Anoth stamp depicts a completely different face, round, merry, and shrewd.

It is chiefly his treatise on algebra, *The Book of Restoration and Equalization,* for which a Khwarizmi is remembered by mathematicians, and it is his chapters on the Hindu system c positional notation by which he is remembered universally, for al-Khwarizmi was th coordinating link between Indian and Arabic mathematics, one of those irreplaceable me capable of facing in two directions at once.

It is al-Khwarizmi's system that we now use, and when he introduced it to his colleague and to the world, in the early part of the ninth century, he told them that it embodied, "th richest, quickest calculation method, the easiest to understand, and the easiest calculatio method to learn." It is supremely useful, he added, "in cases of inheritance, legacie partition, and trade."

All homage, then, to al-Khwarizmi, a man for all symbols and so a man for all seasons.

POSITIONAL NOTATION

The Arabic numerals comprise the nine sensuous shapes **1, 2, 3, 4, 5, 6, 7, 8,** and **9**. Th numerals from one to nine are both basic and primitive—basic because we need somethin with which to start, and primitive because as symbols they cannot be decomposed int anything simpler.

What is yet lacking is a way in which these symbols may be used to denote the numbe beyond nine. Left to their own devices, of course, nine numerals can hardly do more tha name nine numbers.

An especially ingenious Baghdad merchant might have designated the number ten by **9** **1.** The number after that could be designated by **9 + 1 + 1.** This would hardly have been contribution to a merchant's practical concerns, if only because a bill for one hundred ar seventy drachmas would have run on for pages.

The solution to this problem emerged in stages, and it emerged in the way solutions s often emerge in mathematics, a slapped-together strategy followed by a carefully contrive cleanup. Thus merchants with business more pressing than mathematical notation long ag expressed their bills of lading or sale by writing **1 plus X** for the number ten, or simply **1X** where the symbol **1,** by means of its new and unfamiliar *position,* designated the number te and where **X** served simply as a placeholder, a symbol signifying that nothing was bein added to ten.

Thereafter, precisely the same scheme could encompass the numbers that followed, wi the number eleven expressed as **1X plus 1**, or **11**. Having the sense, I am hoping, that h might have discovered something profound, that Baghdad merchant might well have note with satisfaction that by writing **1X** for ten, or **11** for eleven, he had discovered the key t positional notation, the great door swinging open to admit both merchants ar mathematicians.

In any compound numeral of the form **ab**, where **a** and **b** stand in for the numerals from to **9**, position is king and key, indicating *both* that **b** is to be added to **a**, and that **b** mark things in units of one, while **a** marks them in units of ten. That bill of lading now emerge just as that long-dead merchant might have written it:

Ψ

Dates:	17 drachmas
Oil:	13 drachmas
Almonds:	1X drachmas
Figs:	1X drachmas

whereupon positional notation and the ever-useful X come into play again in the total of 5 drachmas.

The placeholder **X** was in time replaced by the symbol **0**, so that **5X** emerges in its moder and familiar form as the numeral **50**.

The cleanup followed years later when the placeholder underwent a further promotion, s that **0** is now regarded as the name of a number in its own right. And for every good reaso The simplest of hygienic routines demands it. The sum of 2 and 0 is 2. Treating **0** as placeholder, and so as a symbol, renders this identity incoherent. A placeholder cannot b *added* to a number, any more than a horse's name can be entered in a race.

Yet the promotion of **0** from a placeholder to the name of a living number is itself hardly model of logical scrupulousness, for if **0** is a name, just *what* does it name? The obviou answer, in virtue of the fact that two plus zero is still two, is that it names *nothing.*

But if zero names nothing, then it is difficult to make sense of *adding* zero to two. There no *adding* nothing to anything.

If, on the other hand, **0** names something, then it is again difficult to see why two plu something should *remain* two. It hardly helps to insist that zero is that unique something th behaves as if it were nothing. Mathematicians have traditionally resolved these difficulties b embracing the thesis that at times something is nothing, a metaphysical achievement th cannot be said to inspire a sense of serenity.

The converse, that at times nothing is something, is, of course, among the most usef declarations of the human race.

In Indian thought there is a connection between zero and *shûnya,* a term meaning th emptiness, the nonexistent, the nonformed, and the noncreated. Curiously enough, zero als seems to have been associated with the infinite, the god Vishnu's foot, and a voyage o water.

In the early years of the nineteenth century, a number of English mathematicians sti regarded zero with unease, and the negative numbers with distaste. Had I been among ther I would have embraced their cause. It may not be too late.

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1 In this book, this rule: When referring *obviously* to symbols, boldface; when referring *obviously* to what they name or designate, regular face; and when there is a latent ambiguity between the two, one disambiguated by context, regular face again. Readers anticipating a completely consistent presentation, I should say at once, are apt to be disappointed. Short of a treatise, there is no way to provide one.

2 Writing in the May 13, 2010, issue of *The New York Review of Books,* and assigning himself a fine eye for detail, John Paulos observed that "whenever I see the bumper sticker 'War is never the answer,' I think that, to the contrary, war most

certainly is the answer, if the question is ‘What is a three-letter word for organized armed conflict?’ ”

But war is, of course, not a three-letter word; it is not a word at all.

3

The urge to get to the bottom of things, if things have a bottom, is not unique to physicists. Why accept the numbers as fundamental if there might be something more fundamental still? Why indeed?

SETS

The introduction of set theory at the end of the nineteenth century persuaded man mathematicians, Bertrand Russell among them, that they had discovered a system by whic the natural numbers could be displaced in favor of something more fundamental. Th creation of Georg Cantor, set theory is the most remarkable single achievement o nineteenth-century mathematics, so much so that David Hilbert was moved to call it paradise. Hilbert was a great mathematician and a careful German prose stylist, his word o choice conveying a very precisely blended mixture of admiration and regret.

Sets are part of a *volkisch* family: troops, tribes, groups, ensembles, even prides, flock rabbles, ribbles, gaggles, and gabbles. These words all are in the end synonyms for *set,* o they depend on the concept of a set for their coherence, a ribble of rabbis, a set of them or collection, but in any case something more than a rabbinical heap. Beyond saying that a set any real or potential object of thought, Cantor very wisely said nothing more.

Sets by their nature have members, things that belong to them. Membership is th fundamental relationship of the theory. No relationship could be more primitive than on answering to the interrogative whether something is in or out.

There is in the idea of a set plainly an extraordinary degree of freedom. This freedo endows even the simplest of sets with a dangerous reproductive urgency. Having for so lon lived lives as sheep, three sheep must now be imagined as a set-theoretical flock, a fact tha mathematicians symbolize by collecting their names in the pen between curled parenthese **{Sheep 1, Sheep 2, Sheep 3}.**

If there were three sheep to begin with, and so three things, there are now *four:* the thre sheep *and* the set collecting them. That set is again an object of thought.

If this is not so, just *what* have we been thinking about?

The universe now includes the set **{{Sheep 1, Sheep 2, Sheep 3}},** whose sole member the set of three sheep whose members in turn are those damned sheep. There are *five* objec in the universe where just a moment ago there were only a handful of sheep.

This is a process that may be indefinitely continued. Sets are profligate; they multiply b iteration. Nor is a uniquely mathematical process at work. It is hardly mathematical at al Commenting on a religious audience in the city of Qom, V. S. Naipaul remarked that "fai like this …," and then because he lacked a word to describe how faithful the faithful were, h added, "faith in the faith," and so suggested that faith, like the operation by which sets ar formed, may be self-applied, faith in the faith being distinct from faith itself.

Something of madness attaches to these iterations, because they suggest no standard o control. Is there for the faithful faith in the faith in the faith?

"Successful indeed are the believers," as the Qur'an observes enigmatically in Sura 23.1.

ONE OF A KIND

The set of sheep has sheep as members; the set of sumo wrestlers, sumo wrestlers; but as fo sheep who are also sumo wrestlers, there are none. The set is empty, thank God. But while is empty, *it* is not nothing.

Quite the contrary. Sets are abstract objects, capable of surviving a radical decline membership. A heap of sheep may be diminished one sheep at a time until both the sheep ar the heap are gone. A heap is nothing more than its sheep. A *set* of sheep, however, surviv sheeplessness to emerge on the other side as a set with no members.

Sets with no members are sisters, as far as set theory goes: nothing belongs to any of ther If two sets are identical because they have the same members, then the empty sets are all th same because none of them has any. Membership hardly comes dearer than this. No club more exclusive. It follows that there is only one empty set, and that is *the* empty set, whic mathematicians designate as ∅, the symbol looking very much like a blindly staring eye, i powers canceled.

Questions about zero now resolve themselves in favor of the artifice that the empty s corresponds to zero. "It's like a party with no guests," a student once remarked.

It is just like that. Merchants are content, and mathematicians too, for without zero there no moving merchandise and certainly no doing mathematics.

If zero corresponds to the empty set, the number one can find purchase as the s containing just the empty set, or {∅}. It has one element, after all. The number two is in th same way identified with the set containing the empty set and the set containing the emp set, making two: 2 = {∅, {∅}}. Thereafter all of the natural numbers may be constructed a tower of *sets* instead of a tower of numbers, so that every natural number corresponds to particular set.

This shows that the numbers may be paired to the sets but hardly that the sets are mo fundamental than the numbers. To see that the set {∅, {∅}} comprises two members, it necessary first to count them. Counting sets is an undertaking as dependent on the natur numbers as counting sheep.

PARADOX

Within Cantor's lifetime, logicians demonstrated that set theory is frankly inconsisten Dangers in mathematics do not get more dangerous than this.

Of the paradoxes, Russell's is the most famous; and it is also the easiest to state. Some se are members of themselves, and some are not. The set of all sets is again a set: it is an obje of thought. But the *set* of all sheep is not a sheep even though it is a set.

What, Russell asked, of the set of all sets that are *not* members of themselves?

Is *it* a member of itself?

It is if it isn't, and it isn't if it is.

This is not a conclusion that anyone might find encouraging, least of all mathematicia otherwise well satisfied that theirs is a discipline in which consistency is championed.

In 1908, Ernst Zermelo proposed a set of axioms for set theory, and suggested hopeful that, as long as they were respected, all would be well.

To this day, no one really knows whether this is so. A *proof* is unavailable.

And all this for zero, which is to say, for nothing.

4

Human knowledge is radically unstable. We are strangers to one another and, more often than not, to ourselves. In telling you that you do not know what you think you know, I am not telling you anything that you do not know.

CERTAINTY

And yet mathematics offers the impression of certainty, a shrug of the absolute. No one raise a skeptical finger in the air on learning that if five is greater than four plus zero, then it also greater than four. It is an impression at odds with human life, and radically at odds wi the other sciences.

Let me offer an example. In the second century a.d., the Greek mathematician ar astronomer Claude Ptolemy created a comprehensive astronomical theory. The heavens h imagined as a great sphere, the earth at its center. Ptolemy's masterpiece is entitled th *Almagest,* the word meaning "the greatest" in Arabic and in the Greek from which the title taken. The *Almagest* is an attempt to grasp the universe in mathematical terms. It is the fir such attempt in history, and for this reason, the greatest, just as its name suggests. Althoug Ptolemaic astronomy is often thought false, Ptolemy and Johannes Kepler are secure in the position as the greatest among the greats. There is no third.

It is in the second part of the first book of the *Almagest,* entitled "On the Order of th Theorems," that Ptolemy describes his ambitions. They are considerable; they are grandios "In the treatise which we propose," he writes, "the first order of business is to grasp th relationship of the earth taken as a whole to the heavens taken as a whole." The second ord of business is to describe "the motions of the sun and the moon," and the third, to give a account of the stars. Ptolemy then summarizes his conclusions. "The heavens [are] spheric in shape and move as a sphere; the earth, too, is sensibly spherical in shape, when taken as whole; in position it lies in the middle of the heavens, very much like its center; in its siz and distance, it has the ratio of a point to the sphere of the fixed stars, and it has no motio from place to place."

"Absolutely all phenomena," Ptolemy adds somberly, "are in contradiction to any of th alternative notions that have been proposed."

NOTHING HUMAN IS CERTAIN

For fifteen hundred years, the *Almagest* appeared as solid and as enduring as pig iron, th theory that it expressed brilliantly meeting the demands of new astronomical data, such a the retrograde motion of the planets, by means of an elaborate system of epicycles ar deferents. Until well into the seventeenth century, the advantages of the Copernican syste were not clear, and its disadvantages considerable. Copernican astronomers could n plausibly explain why, if the earth were in motion around the sun, no one on its surface eve noticed.

And yet Ptolemaic astronomy was discarded soon afterward, and thereafter ridiculed fo the very techniques that lent it accuracy. It is today an object lesson, and so a warning, th facts in the end turning against the theory:

—The earth is not at the center of the solar system.

—The sun does not move.

—The planets do not describe circles in the sky.

—The heavens are not spherical.

Nihil homini certum est, as Ovid observed. Nothing human is certain.

AN EXCEPTION TO OVID

Mathematics is the great exception to this melancholy observation; and Ptolemy is, in th regard, a one-man multitude. A number of powerful theorems in geometry rise from the rui of his system, like tendrils forcing themselves through the rubble and toward the sun.

A circle is given in the plane and within the circle a four-sided figure inscribed. The sum the products of their sides, Ptolemy demonstrated, is equal to the product of the tw diagonals. This result is today known as Ptolemy's theorem.

The contrast between the man of astronomy and the man of mathematics is poignan Ptolemy had invested his hope for glory in his theory of the heavens; the mathematics that developed, he regarded as an instrument.

The glory was there all along, but it was not where Ptolemy had thought to find it.

What accounts for the difference between the man of mathematics and the man of science

The popular view is that in mathematics proof is possible, and beyond mathematics it not. "As far as the laws of mathematics refer to reality," Einstein remarked, "they are n certain, and as far as they are certain, they do not refer to reality." Of this view, the mo that can be said is that it is bizarre. We take reality with our eyes wide open. Why shou what we take in with our senses be *less* certain than what we conjure up in saying that, if fou is greater than three and three greater than two, then four is greater than two? It would see to be quite the other way around.

No doubt there are proofs in mathematics. Proofs are the mathematician's coin in trad The question is why there are none *elsewhere.*

A proof, after all, is a mathematical *argument,* and so a part of an old and familiar huma genre. The logician, and not the mathematician, is in charge. It is the opportunity of lifetime. Beyond mathematics, the logician has, in any case, a very substantial portfolio. Sin his business is conveying premises correctly to their conclusions, his subject is relevant to a activity that arrays human beings against one another or against themselves: domest disputes; financial controversies; conflicts about abortion, family life, corporate organizatio international law, simple decency, flag burning, homeopathic medicine, ancient archaeolog women's rights, the rules of warfare, dress codes, intelligent design, conspiracy theorie Freudian psychology, or anything else that may be plausibly embedded in a continuu ranging from *Honey, Let's Not Fight Anymore* to *Exterminate All the Brutes.*

And yet, across the vast range of arguments offered, assessed, embraced, deferred, delaye or defeated, it is only within mathematics that arguments achieve the power to comp allegiance because they are seen to command assent.

No philosophical theory has ever shown why this should be so. It is a part of the myste of mathematics.

Aristotle was the first and the greatest of logicians, with even Kurt Gödel judged great (by Robert Oppenheimer, among others) because he was the greatest logician *since Aristotl* Entitled the *Organon,* Aristotle's work on logic is virtually the only part of his remarkab corpus to be reliably traced back to his own hand.

The eighteenth-century English philosopher Thomas Reid has offered a shrewd account Aristotle's genius. Aristotle, Reid notes, “had very uncommon advantages.” He was born Greece “in an age when the philosophical spirit in Greece had long flourished,” and was fo “twenty years a favourite scholar of Plato, and tutor to Alexander the Great, who bo honoured him with his friendship, and supplied him with every thing necessary for th prosecution of his inquiries.” These advantages, Reid goes on to say, “he improved b indefatigable study, and immense reading.” And as to his genius, “it would be disrespectful mankind not to allow an uncommon share to a man who governed the opinions of the mo enlightened part of the species near two thousand years.” This is both handsome as a tribu and correct as a compliment. Determined to find some criticism to offer Aristotle, the sobe Reid can think to say only that he was human, for he “seems to have had,” Reid argues, “ greater passion for fame than for truth, and to have wanted rather to be admired as th prince of philosophers than to be useful.”

WHAT IF, WHAT THEN

To Aristotle is due the foundational insights of logic itself, the keys to its nature. There a just two. An argument is valid in virtue of its form and not its content; and the validity of a argument is conditional and so a matter of asking *what if* and seeing thereafter *what then.*

Here is an argument offered by the American logician Alonzo Church (in the introductio to his treatise, *Introduction to Mathematical Logic*):

The first premise:	Brothers have the same surname.
The second:	Richard and Stanley are brothers.
The third:	Stanley has the surname Thompson.
The conclusion:	Richard has the surname Thompson.

And here, Church adds, is quite another:

The first premise:	Complex numbers with real positive ratios have the same amplitude.
The second:	$i - \sqrt{3}/3$ and ω are complex numbers with real positive ratios.
The third:	ω has amplitude $2\pi/3$.
The conclusion:	$i - \sqrt{3}/3$ has amplitude $2\pi/3$.

In both arguments, premises are passing toward their conclusions, but dealing as it do with a branch of mathematics known as complex analysis, the second argument might we