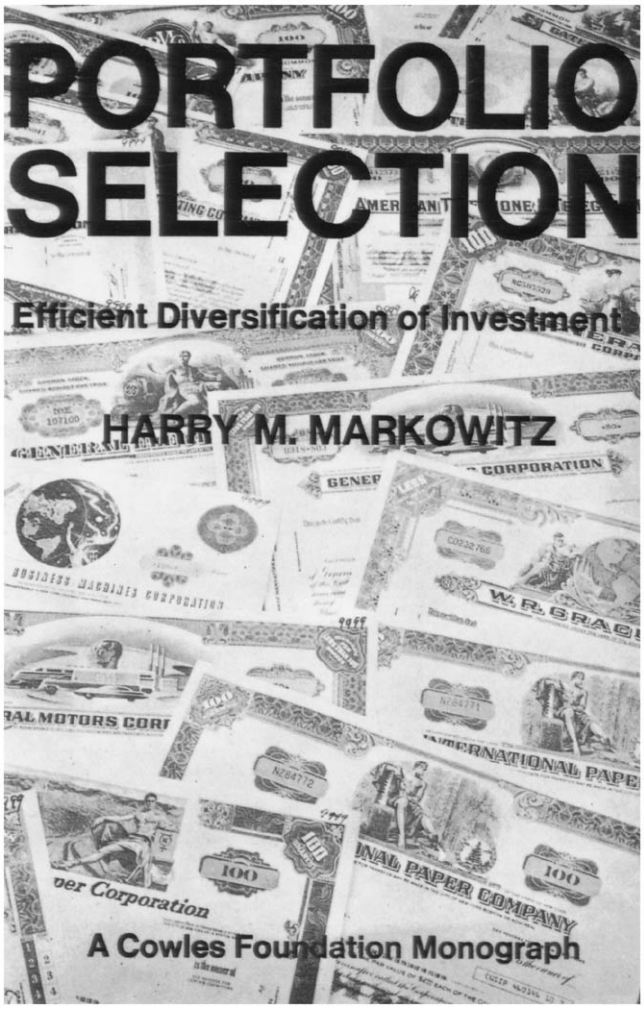
PORTFOLIO SELECTION
Efficient Diversification of Investment
HARRY M. MARKOWITZ
A Cowles Foundation Monograph

COWLES FOUNDATION
for Research in Economics at Yale University

MONOGRAPH 16

COWLES FOUNDATION

For Research in Economics at Yale University

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A list of Cowles Foundation Monographs appears at the end of this volume.

Portfolio Selection

EFFICIENT DIVERSIFICATION OF INVESTMENTS

Harry M. Markowitz

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To Mildred and Morris

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PREFACE TO THE SECOND PRINTING

The correction of a number of errors originally missed in proofreading and the addition of a new bibliography differentiate this printing from the first.

Most of the errors were discovered while this book was being translated into Japanese by Professor Suzuki of Tokyo University, under the auspices of the Yamaichi Investment Trust Management Co., Ltd. I am greatly indebted to Professor Suzuki for these corrections, and to Tatsuo Majima, President, and Kazuo Kitamura, Director and Manager of the Research Department of Yamaichi, with whom I corresponded concerning the arrangements for, and the progress of, the translation.

The new bibliography (pp. 308–15) was graciously supplied by Mark Rubinstein. It was drawn from a larger bibliography of his which is to appear elsewhere.[1] Rubinstein's annotated condensed bibliography is included here to provide the reader with an introduction to the extensive recent literature on portfolio selection.

This literature includes works which apply the "Expected return, Variance of return (E, V) efficient set analysis" to areas beyond those covered in this book, and works which attack or defend the E, V efficient set approach. A brief characterization of what, to my mind, has been the impact of this literature on the contents of this book may be of interest to some readers. I will not try to argue my present positions here but will merely state them. Not all contributions of substantial significance are even mentioned in this thumbnail sketch. My comments are organized according to the chapters of the book.

Chapter 6, "Return in the Long Run," serves to illustrate the method of quadratic approximation which has practical and theoretical value in conjunction with portfolio analysis. While writing the book at the Cowles Foundation in 1955–56, I concluded (as essentially did Latané[2] independently) that the investor who is currently reinvesting everything for "the long run" should maximize the expected value of the logarithm of wealth. Mossin[3] and Samuelson[4] have each shown that this conclusion is not true for a wide range of functions relating utility to wealth at the end of the last time period, T. The fascinating Mossin-Samuelson result, combined with the straightforward argu-

1. R. Bruce Ricks and Mark E. Rubinstein, eds., *Portfolio Theory: Advanced Readings*, forthcoming from McGraw-Hill Book Company, New York.
2. Henry A. Latané, "Criteria for Choice among Risky Ventures," *Journal of Political Economy*, April 1959.
3. Jan Mossin, "Optimal Multiperiod Portfolio Policies," *Journal of Business*, April 1968.
4. Paul A. Samuelson, "Risk and Uncertainty: A Fallacy of Large Numbers," *Scientia*, April-May 1963.

ments supporting the earlier conclusions, seemed paradoxical at first. I have since returned to the view of Chapter 6 (concluding that: for large T, the Mossin-Samuelson man acts absurdly, like a player who would pay an unlimited amount for the St. Petersburg game or its Cramer generalization; the terminal utility function must be bounded to avoid this absurdity; and the argument in Chapter 6 applies when utility of terminal wealth is bounded).

Be this as it may, the reader should not lose sight of the illustration in Chapter 6 of the method of quadratic approximation. I will return to this later in connection with Chapter 13.

Chapter 7, which presents the geometrical analysis of how E, V efficient sets "look," contains an erroneous footnote on how to include short sales. The procedure in the footnote implies that funds are released by a short sale rather than that collatoral is required. The reader experienced with linear programming models should have no difficulty in supplying the proper construction. In class, in conjunction with Chapters 7 and 8, I illustrate a variety of concrete portfolio requirements which can be handled as special cases of the general constraints $Ax = b$, $x \geqq 0$ (matrix notation). The proper treatment of the short sale case is included.

Chapter 8 describes the critical line method for generating E, V efficient portfolios for any constraint set of the form $Ax = b$, $x \geqq 0$. An up-to-date treatment of this topic would also discuss the possibility of special algorithms when the covariance matrix can be simplified as in the diagonalized models of Sharpe[5] and of Cohen and Pogue.[6] Much of the interest in, and experimentation with, portfolio analysis in the 1960s was stimulated by the publication of Sharpe's article and the availability of his single-index, FORTRAN, portfolio selection program.

Chapter 9 presents the geometric analysis and computing procedures if semi-variance is substituted for variance in the efficient set analysis. The algorithm is more costly but not prohibitive. While semi-variance has adherents in principle, no computer code has been produced as yet.

Part IV of the book (Chapters 10 thru 13) presents philosophical justifications and limitations of the E, V efficient set analysis. Chapter 12 presents my own first principles, based on those of Leonard J. Savage,[7] concerning rational behavior over time under uncertainty. Chapters 10 and 11 deal with special cases of Chapter 12, presented as expository stepping-stones toward the more general theory. Chapter 13 discusses additional conditions sufficient for the relatively inexpensive E, V efficient set analysis to give at least near-optimum results.

The most discussed theoretical objections to the E, V efficient set analysis

5. William F. Sharpe, "A Simplified Model for Portfolio Analysis," *Management Science*, January 1963.

6. Kalman J. Cohen and Jerry A. Pogue, "An Empirical Evaluation of Alternative Portfolio-Selection Models," *Journal of Business*, April 1967.

7. Leonard J. Savage, *The Foundations of Statistics* [32].

seem to be: (1) objections to the expected utility maxim (e.g. Hirshleifer[8]), (2) the assertion that the probability distribution of returns has an infinite variance (Mandelbrot[9] and Fama[10]), and (3) objections to the principle of quadratic approximation (Pratt-Schlaifer[11] and Borch[12]). My own chief concern lies elsewhere.

(1) My view of the expected utility maxim is the same as when I wrote Part IV. I believe that the reply to Hirshleifer's objections is already in Part IV, particularly Chapter 11. Parenthetically, I know of no plausible axiom system implying probability beliefs that does not also imply expected utility.

(2) Regarding the Mandelbrot-Fama contention that variance is infinite: (a) I am willing to assume that all my subjective distributions of return are bounded—e.g. between 100 percent loss and a trillion percent gain—and therefore have all their moments. (b) The strange conclusion that variance is infinite is derived by starting with the assumption that the probability distribution of hour-to-hour fluctuations in security prices has the "same form" as, say, that of month-to-month fluctuations, which in turn has the same form as, say, the probability distribution of year-to-year fluctuations. This assumption seems less than certain when we contrast the business determinants of the year-to-year fortunes of an enterprise with the market determinants of the hour-to-hour fluctuations in its stock. The assumption becomes even more questionable when we learn that the assumption implies a priori that either the distribution is normal or it has infinite variance—excluding not only all bounded distributions, but also most of the familiar unbounded distributions such as χ^2 and Student. Having assumed this much, the next step is to infer empirically that since the distribution is not precisely normal, it must have infinite variance.

(3) In light of the results of Pratt-Schlaifer and Borch regarding quadratic utility functions, the argument concerning quadratic approximation (in Chapters 6 and 13) should be modified. In these chapters I show how to estimate expected utility, given only E and V, by formulae derived from either (a) a quadratic approximation to the utility function around $X_0 = 0$, or (b) the

8. Jack Hirshleifer, "State Preference, Risk Aversion, and the Utility-of-Income Function," Part I of "Investment Decision under Uncertainty: Application of the State-Preference Approach," *Quarterly Journal of Economics*, May 1966.

9. Benoit Mandelbrot, "Stable Paretian Random Fluctuations and Multiplicative Variation of Income," *Econometrica*, October 1961; and Benoit Mandelbrot and H. M. Taylor, "On the Distribution of Stock Price Differentials," *Operations Research*, November-December 1967.

10. Eugene F. Fama, "Mandelbrot and the Stable Paretian Hypothesis," *Journal of Business*, October 1963; and idem, "The Behavior of Stock Market Prices," *Journal of Business*, January 1965.

11. John W. Pratt, "Risk Aversion in the Small and in the Large," *Econometrica*, January-April 1964. Professor Pratt wrote me: "The history of this [quadratic utility function] business has always bothered me. The original idea was [Robert] Schlaifer's, as expressed in his 11/31/61 memo. He proved necessity. I proved sufficiency and various properties soon thereafter."

12. Karl Borch, "A Note on Uncertainty and Indifference Curves," *Review of Economic Studies*, January 1969.

same around $X_0 = E$. In empirical tests, such as the illustrative experiment in Chapter 6, it repeatedly turns out that method (b) does better than (a). It now further turns out that method (a) is subject to the Pratt-Schlaifer and Borch objections, while method (b) is not. Thus—until some still better method presents itself for estimating expected utility from the E and V of "realistic" distributions—the use of the method of quadratic approximation, in theory and practice, should be confined to method (b) with $X_0 = E$.

My own chief theoretical worry remains, as in Chapter 13, with the gap between a theory based on perfect liquidity and the existence of illiquidities. In practice the gap is filled with plausible but ad hoc approximations. Keith Smith's[13] simulation analyses of portfolio adjustment procedures are important steps in the right directions, but this critical area remains mostly unexplored.

As noted above, this brief sketch of my current general position does not attempt to mention every important recent contribution to portfolio theory and its application in practice.

H.M.M.

Beverly Hills, California
August, 1970

13. Keith V. Smith, "Alternative Procedures for Revising Investment Portfolios," *Journal of Financial and Quantitative Analysis*, December 1968.

PREFACE

This monograph presents techniques for the analysis of portfolios of securities. Although the techniques are mathematical in nature, the monograph is written primarily with the non-mathematician in mind. Part I discusses and illustrates the inputs, outputs, and objectives of a formal portfolio analysis. Part II presents concepts and theorems needed subsequently in our exposition. Part III uses the prerequisites developed in Part II to go more deeply into techniques of portfolio analysis. Part IV, finally, discusses the theory of rational behavior and its applications to the selection of portfolios.

The appendices of the book are for the mathematically trained reader only. Their main function is to prove certain more advanced relationships noted and used in the text.

The mathematically trained reader may find the following suggestions helpful: Part I should be read by way of motivation and illustration. Part II may be skimmed or skipped. It attempts an elementary exposition of the minimum requirements for the rest of the text. (Within Part II, Chapter 3 culminates in the formula for the expected value of a linear combination of variables. Chapter 4 culminates in the formula for the variance of a linear combination of random variables. Chapters 5 and 6 present certain laws of large numbers.)

Part III presents a geometric analysis of, and computing procedures for, the derivation of efficient sets. Appendices A and B demonstrate that the computing procedures presented in Chapters 8 and 9 do, in fact, produce the desired efficient sets. Part IV, on the theory of rational behavior, presents and applies the utility and personal probability maxims. Appendix C continues the text's discussion of axiom systems for expected utility.

I am indebted to several organizations for aid and encouragement. The method of analysis presented in this monograph was originally developed for my doctoral dissertation. This early work (1950–51) was supported by the Social Science Research Council and the Cowles Commission for Research in Economics. From August 1955 through May 1956, while on leave from the RAND Corporation, I did most of the writing required to transform thesis into monograph. During this period I was located at the Cowles Foundation for Research in Economics at Yale, where the writing of the monograph was made possible by a grant from the Merrill Foundation for Advancement of Financial Knowledge. To these organizations I wish to express my gratitude for intellectual and material support.

I am also indebted to many individuals. James Tobin and Roy Radner read one or more drafts of many chapters and provided valuable advice with

respect to both content and exposition. Mrs. Markowitz read drafts of several chapters and provided helpful suggestions concerning exposition for the non-mathematician.

The content of Part IV of this book reflects a series of conversations with Gerard Debreu. The content of Part I reflects sessions with Horace F. Isleib, Investment Officer of Yale University, and Ralph W. Halsey, Jr., Assistant Investment Officer.

This monograph benefited from the diligence of a number of people: Ewing Jackson Webb, who prepared the inputs to the ten-security example; Harold Watts, Robert Z. Aliber, and Leroy S. Wehrle, who proofed the final draft for the Cowles Foundation; Mrs. Natalie Sirkin, who did the hard part of preparing the bibliography; and Miss Althea Strauss, who efficiently supervised the typing of two or three drafts of each chapter.

While the afore-mentioned individuals and organizations have aided immeasurably in the writing of this book, all opinions and any errors contained herein are, of course, my own responsibility.

HARRY M. MARKOWITZ

New York City
February, 1959

PART I

INTRODUCTION AND ILLUSTRATIONS

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CHAPTER I

INTRODUCTION

THE ANALYSIS OF PORTFOLIOS

This monograph is concerned with the analysis of portfolios containing large numbers of securities. Throughout we speak of "portfolio selection" rather than "security selection." A good portfolio is more than a long list of good stocks and bonds. It is a balanced whole, providing the investor with protections and opportunities with respect to a wide range of contingencies. The investor should build toward an integrated portfolio which best suits his needs. This monograph presents techniques of Portfolio Analysis directed toward determining a most suitable portfolio for the large private or institutional investor.

A portfolio analysis starts with information concerning individual securities. It ends with conclusions concerning portfolios as a whole. The purpose of the analysis is to find portfolios which best meet the objectives of the investor.

Various types of information concerning securities can be used as the raw material of a portfolio analysis. One source of information is the past performance of individual securities. A second source of information is the beliefs of one or more security analysts concerning future performances. When past performances of securities are used as inputs, the outputs of the analysis are portfolios which performed particularly well in the past. When beliefs of security analysts are used as inputs, the outputs of the analysis are the implications of these beliefs for better and worse portfolios.

This introductory chapter discusses broad principles upon which the techniques of portfolio analysis are based. The next chapter discusses the inputs, outputs, and objectives of illustrative portfolio analyses. Subsequent parts of the monograph go more deeply into the techniques by which information concerning securities is transformed into conclusions concerning portfolios.

The Uncertainty of Security Returns

Uncertainty is a salient feature of security investment. Economic forces are not understood well enough for predictions to be beyond doubt or error. Even if the consequences of economic conditions were understood perfectly, non-economic influences can change the course of general prosperity, the level of the market, or the success of a particular security. The health of the President, changes in international tensions, increases or decreases in military spending, an extremely dry summer, the success of an invention, the miscalculation of a business management—all can affect the capital gains or dividends of one or many securities.

We are expecting too much if we require the security analyst to predict with certainty whether a typical security will increase or decrease in value. Even if he could assemble all information, including information available only to the managers of the corporation and information available only to its competitors, the security analyst might still be forced to conclusions such as:

> This security may be expected to do well if securities in general do well. It must be expected to do poorly if securities in general do poorly. Even this following of the market is not certain. There are weaknesses which may cause it to do poorly even though securities in general are performing well: The possibility of a labor dispute or of an aggressive competitor cannot be ignored. On the other hand, there are potentialities which may bring success greater than even the corporation management dares hope. The new styling of the product, the (not inexpensive) advertising campaign, and the expansion of production facilities may prove to be a magic combination, fulfilling all expectations for it.

Only the clairvoyant could hope to predict with certainty. Clairvoyant analysts have no need for the techniques of this monograph.

The existence of uncertainty does not mean that careful security analyses are valueless. The security analyst may be expected to arrive at reasonable opinions to the effect that:

> The return (including capital gains and dividends) on security A is less uncertain than that on security B; the return on security C is more closely connected to the course of the general market than is that on security D; the growth of security E is more certain but has less potential than that of security F; only if the demand for their industry's product continues to expand (as it is likely, but not certain, to do) will the return on securities G and H be satisfactory.

Carefully and expertly formed judgments concerning the potentialities and weaknesses of securities form the best basis upon which to analyze portfolios.

Correlation among Security Returns

A second salient feature of security investment is the correlation among security returns. Like most economic quantities, the returns on securities tend to move up and down together. This correlation is not perfect: individual securities and entire industries have at times moved against the general flow of prosperity. On the whole, however, economic good and ill tend to spread, causing periods of generally high or generally low economic activity.

If security returns were not correlated, diversification could eliminate risk. It would be like flipping a large number of coins: we cannot predict with confidence the outcome of a single flip; but if a great many coins are flipped we can be virtually sure that heads will appear on approximately one-half of them. Such canceling out of chance events provides stability to the disbursements of insurance companies. Correlations among security returns, however, prevent a similar canceling out of highs and lows within the security market. It is somewhat as if 100 coins, about to be flipped, agreed among themselves to fall, heads or tails, exactly as the first coin falls. In this case there is perfect correlation among outcomes. The average outcome of the 100 flips is no more certain than the outcome of a single flip. If correlation among security returns were "perfect"—if returns on all securities moved up and down together in perfect unison—diversification could do nothing to eliminate risk. The fact that security returns are highly correlated, but not perfectly correlated, implies that diversification can reduce risk but not eliminate it.

The correlation among returns is not the same for all securities. We generally expect the returns on a security to be more correlated with those in the same industry than those of unrelated industries. Business connections among corporations, the fact that they service the same area, a common dependence on military expenditures, building activity, or the weather can increase the tendency of particular returns to move up and down together.

To reduce risk it is necessary to avoid a portfolio whose securities are all highly correlated with each other. One hundred securities whose returns rise and fall in near unison afford little more protection than the uncertain return of a single security.

Objectives of a Portfolio Analysis

It is impossible to derive all possible conclusions concerning portfolios. A portfolio analysis must be based on criteria which serve as a guide to the important and unimportant, the relevant and irrelevant.

The proper choice of criteria depends on the nature of the investor. For some investors, taxes are a prime consideration; for others, such as non-profit corporations, they are irrelevant. Institutional considerations, legal restrictions, relationships between portfolio returns and the cost of living may be important to one investor and not to another. For each type of investor the details of the portfolio analysis must be suitably selected.

Two objectives, however, are common to all investors for which the techniques of this monograph are designed:

1. They want "return" to be high. The appropriate definition of "return" may vary from investor to investor. But, in whatever sense is appropriate, they prefer more of it to less of it.
2. They want this return to be dependable, stable, not subject to uncertainty. No doubt there are security purchasers who prefer uncertainty, like bettors at a horse race who pay to take chances. The techniques in this monograph are not for such speculators. The techniques are for the investor who, other things being equal, prefers certainty to uncertainty.

The portfolio with highest "likely return" is not necessarily the one with least "uncertainty of return."[1] The most reliable portfolio with an extremely high likely return may be subject to an unacceptably high degree of uncertainty. The portfolio with the least uncertainty may have an undesirably small "likely return." Between these extremes would lie portfolios with varying degrees of likely return and uncertainty.

If portfolio A has both a higher likely return and a lower uncertainty of return than portfolio B and meets the other requirements of the investor, it is clearly better than portfolio B. Portfolio B may be eliminated from consideration, since it yields less return with greater uncertainty than does another available portfolio. We refer to portfolio B as "inefficient." After eliminating all such inefficient portfolios—all such portfolios which are clearly inferior to other available portfolios—we are left with portfolios which we shall refer to as "efficient." These consist of: the portfolio with less uncertainty than any other with a 6% likely return, the portfolio with less uncertainty than any other with a 7% likely return, and so on. It cannot be said of two efficient portfolios "the first is clearly better than the second since it has a larger likely return and less uncertainty." All such cases have been eliminated.

The proper choice among efficient portfolios depends on the willingness and ability of the investor to assume risk. If safety is of extreme importance, "likely return" must be sacrificed to decrease uncertainty. If a

[1] In later chapters we must give precise definitions to terms such as "likely" and "uncertainty." For the present we may leave them as rough, intuitive concepts.

greater degree of uncertainty can be borne, a greater level of likely return can be obtained. An analysis of the type presented in this monograph:

first, separates efficient from inefficient portfolios;

second, portrays the combinations of likely return and uncertainty of return available from efficient portfolios;

third, has the investor or investment manager carefully select the combination of likely return and uncertainty that best suits his needs; and

fourth, determines the portfolio which provides this most suitable combination of risk and return.

CHAPTER II

ILLUSTRATIVE PORTFOLIO ANALYSES

INPUTS TO AN ILLUSTRATIVE PORTFOLIO ANALYSIS

The nature and objectives of portfolio analyses may be illustrated by a small example concerned with portfolios made of one or more of nine common stocks and cash. The nine securities, listed in Figures 1a to 1i,

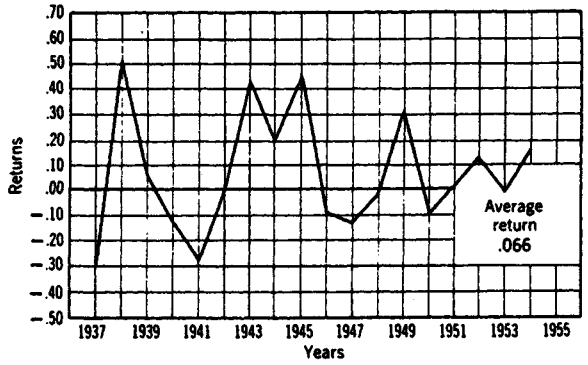

Figure 1a. Returns on security 1, American Tobacco, Common.

include a utility, a railroad, a large and a small steel company, and several other manufacturing corporations. Cash is included in the analysis as a tenth "security." No special significance should be attached to this list of securities other than that it will be used in illustrating principles of portfolio analysis.

An actual portfolio analysis would start from a much longer list of promising securities. Not all these securities would appear in the final desirable portfolio. They enter the analysis as candidates for a place in the desirable portfolio.

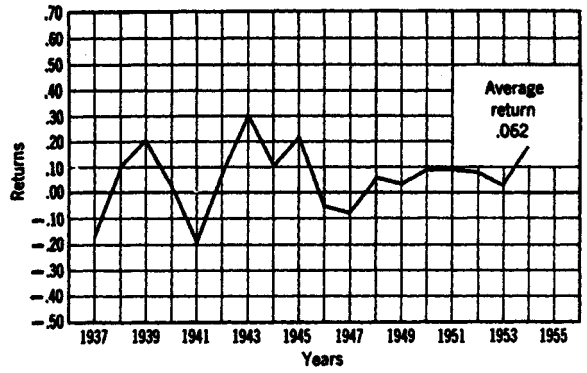

Figure 1b. Returns on security 2, American Tel. & Tel., Common.

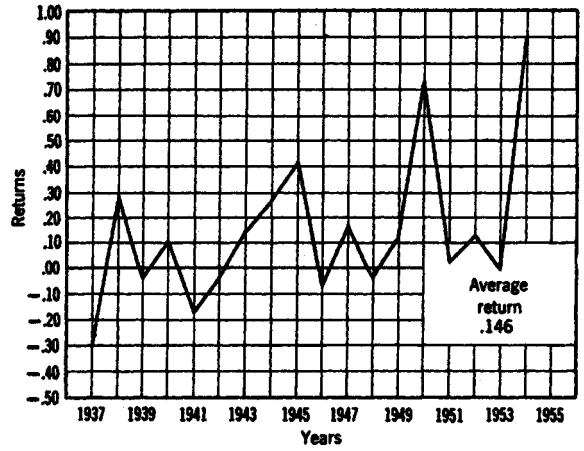

Figure 1c. Returns on security 3, United States Steel, Common.