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Stephen Bruce Sontz

Principal Bundles

The Classical Case

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ISSN 0172-5939 ISSN 2191-6675 (electronic)
Universitext
ISBN 978-3-319-14764-2 ISBN 978-3-319-14765-9 (eBook)
DOI 10.1007/978-3-319-14765-9

Library of Congress Control Number: 2015934141

Mathematics Subject Classification (2010): 14D21, 53C05, 55R10, 58A05, 78A25, 81T13

Springer Cham Heidelberg New York Dordrecht London

Printed on acid-free paper

Springer International Publishing AG Switzerland is part of Springer Science+Business Media (www.springer.com)

A Lilia con muchísimo amor

Preface

Yet another text on differential geometry! But *why*? The answer is because this book is focused on one particular topic in differential geometry, that is, principal bundles. And the aim is to get the reader to an understanding of that topic as efficiently as possible without oversimplifying its foundations in differential geometry. So I aim for an early arrival at the applications in physics that give this topic so much of its flavor and vitality. But this is just the view from the classical perspective, which is why we speak of classical differential geometry (sometimes simply classical geometry) and refer to the topic of this volume as classical principal bundles.

There is a saying that all is prologue. As with any saying, it has a limited range of applicability. But here it is relevant since the topic of this volume serves as preparation for the corresponding quantum (or noncommutative) case, which will be the topic of the companion volume [44]. That is usually known as noncommutative geometry, and the corresponding bundles are known as quantum principal bundles. That is a newer field still being studied and refined by contemporary researchers. While none of this quantum theory has reached a stage of general consensus on "what it is all about" or on what is the "best" approach, we do know enough about the various approaches to be sure that certain common elements will dominate future research. And a lot of the intuition and motivation for the quantum theory comes from the classical theory presented in this volume, which serves as prologue.

Those are the goals of the two volumes. And as far as I am aware, these two texts together comprise the only published books devoted exclusively to the exposition of principal bundles in these two settings. That's *why*.

For *whom* are these books intended? The intended audience for these two volumes are folks who have some interest in learning topics of current interest in geometry and their relation with mathematical physics. I have basically two groups in mind.

The first group comprises mathematicians who have not seen the applications of principal bundles in physics. For them, the first part of this book should be more accessible since concepts are defined and theorems are proved, all according to the modern criteria of mathematical rigor. The second part of this book might require more effort for them since some amount of physical intuition is always helpful for understanding the applications.

The second group is physicists with some familiarity with the standard topics in physics (such as classical electrodynamics) but who have not seen how the mathematics used for those topics works in detail as an aspect of geometry. For this group, the first part of the book will likely require more work. So I have tried to include a lot of motivation and intuition to make that part more accessible for them. Consequently, my more mathematically inclined readers may at times find too many details for their level of expertise. Please be patient and understand why this is happening. In short, this book is meant to help bridge the communication gap between the two communities of physicists and mathematicians. In brief, that's for *whom*.

And *how*? I have tried to make the presentation throughout the book as intuitive as I possibly could. Ideas and intuitions are emphasized as being important elements in formulating the theory. I even think that intuition and ideas are more important than rigor, though they should not exclude rigor but rather precede it to give motivation. This is the "tricky bit," as the saying goes. But intuition is not a sufficient condition for getting things right, nor is rigor. Intuition in physics has led even Nobel laureates to arrive at ideas in contradiction to experiment. As an example, there were those who rejected the initial conjecture of parity being violated in the weak interactions. And mathematicians, even armed with rigor, have also fallen into error. The original "proof" of the four-color theorem comes to mind. Nobody is perfect!

Also, there are plenty of exercises to keep the reader active in the creation process that is essential to understanding, as Feynman so neatly puts it in the quote at the start of Chapter 1. The intention is that the exercises should give the reader hands-on experience with the ideas and intuitions. Some exercises are rather routine, while others are meant to challenge the reader. And I do not even tell you which ones are the routine exercises and which ones are the tough nuts to crack. All of that recognition is part of your own learning process. However, there is an appendix with further discussion of the exercises, including hints. I recommend that you hold off on looking at this (as well as the multitude of other texts on these topics) for as long as possible. If not longer. Even so, some problems may remain quite difficult. But this manner of presentation is intentional and is meant to help the reader in the long run.

I have been accused of asking the reader to write major parts of the book by doing difficult exercises with no hints or suggestions in the text itself. Well, that is true. The more of this book that you, my kind reader, can write, the better off you will be. And sometimes I even give an exercise before acquainting you with the usual tools required for solving it. So gratification is not always immediate. Welcome to the real world of science! This is meant to be a difficult book, much more so than the usual introductory technical texts on the market. That's *how*.

I have avoided a strictly historical approach since the way we have arrived at this theory can obscure its logical structure. For example, the Dirac monopole appeared in the 1930s but is presented here near the end of the book as a special case of the theory developed decades later.

Mathematicians should be aware that physics is a discipline with its own logical structure, though that is not always clear to mathematicians. Ideas flow into other ideas. Results that involve very particular ideas (such as the Yang–Mills theory) then motivate generalizations, which in turn illuminate the original particular case (among other examples) in new ways. This is not the paradigm of definition, theorem, proof. Nonetheless, it is a logical structure in its own right. In the sections on applications to physics, I have tried to include a lot more than the usual amount of motivation based on physics ideas in the hope of helping my mathematically inclined readers. The physicists among my readers might find this boring, and so I kindly request their patience and understanding. But they might be more challenged by the translation of these ideas into a mathematically rigorous geometric formulation.

These are texts meant for learning the material, for either actual students or others who want to learn about principal bundles. The pace is designed for use in a course or for self-study. My experience is that there is enough material here for a one-year graduate course although undergraduates with an adequate background and motivation could profit from such a course.

These are not definitive treatises meant only for the purpose of giving experts a place to look up all the variants of theorems. The experts should not expect too much from either of these volumes, except perhaps as a way of organizing these topics for their own courses. I have not included a multitude of fascinating topics, both in geometry and in physics. These introductory texts should serve to motivate the reader to carry on with study and research in what is *not* presented as well as in what is presented.

The prerequisites for this volume consist of a bit of many things, such as the basic vocabulary of group theory (not to be confused with group therapy); a smattering of linear algebra, including tensor products (even though this will be briefly reviewed); multivariable calculus of real variables, including vector calculus notation; at least a vague appreciation of what a nonlinear differential equation is; some talking points from elementary topology (such as compact, open, closed, Hausdorff, continuous, etc.); and something about categories as a system of notation and diagrams, but not as a theory. And a bit of physics for the chapters with such applications may be useful though I have tried to keep that material as self-contained as possible.

More than anything else, the present volume is my own personal take on classical differential geometry, the role that principal bundles play in it, and how all this relates to physics. Since the goal of the book, as its title reveals, is the exposition of the theory of principal bundles in the classical case, not all of the quite fascinating topics of differential geometry will be presented in the first chapters, but more than enough to get us to that goal. But to understand what principal bundles are “good for” requires meaningful examples as well. And since my personal motivation comes from physics, I devote the remainder of the book to several chapters on examples taken from physics. These chapters make this book more than just another introduction to principal bundles. I tried to make these as self-contained in terms of the physics content as I could, but I rely heavily on the mathematical theory developed earlier in the book. I have always felt that learning physics with little more than a solid mathematics background is far easier than the other way around. So I beseech my physicist readers not to despair during the first part of this book. The trip may be tougher than you'd like, but the payoff should be well worth it.

Nothing here is original. I can merely hope that my way of presenting this material has an appealing style that makes things intuitive and accessible. This book is my own personal response to the challenge Richard Feynman posed to himself, and to no one else, in the famous last blackboard quote cited in Chapter 1.

This is also the place to give thanks to all those who helped me create my own understanding of these topics. It turns out to be the sum of many contributions, some small, some quite large. And all over a long period of time. Many are people whom I have never met but only know through their publications. Names like Courant and Robbins come to mind since they were the first to show me how to think about mathematics in their book [5]. Others have spoken directly to me, but about topics that seem to be worlds away from differential geometry. A name like Larry Thomas comes to mind since he was the first to show me how to think about mathematical research. Also, when I was a graduate student at Virginia, David Brydges gave a very pretty course about gauge theory that helped me put a lot of details that I vaguely knew into sharper focus. The list goes on and on. Please, my friends and colleagues, do not be offended that you are not explicitly mentioned here. The list is way too long.

Also, I gratefully thank all those at Springer who produced a book out of a manuscript. Among those, special thanks go to Donna Chernyk, my editor. The comments of the anonymous referees also helped me improve the book. For those not aware of the details of this conversion process, let me say that it is neither a function nor a functor as far as I can figure out, but it is certainly a lot of work.

This volume is based on several introductory graduate courses that I have taught over the years. Among the participants in those courses, very special thanks are due to Claudio Pita for his helpful comments and continual insistence on ever-clearer explanations.

But there is one person who ignited the spark that made differential geometry quite clear from the very start. And that is Arunas Liulevicius. I most graciously thank him. He taught the first courses I ever took on differential geometry. His intelligent, rigorous discourse, blended with an authentic, bemused humor, made a great impact on me as a model of how to approach mathematics in general, not just these topics. And I never had to un-learn anything—which is a sort of litmus test in itself too! In terms of the specific topics in the beginning of this book, I am more indebted to him than to anyone else.

As for errors, omissions, and all other sorts of academic misdemeanors, there is no one to blame but myself. Please believe me that though I be guilty of whatever shortcoming, I am innocent of any malicious intent. In particular, omissions in the references are mere reflections of my limited knowledge. I request my kind reader to help me out with a message to set me right. I will be most appreciative.

Guanajuato, Mexico
September 2014

Stephen Bruce Sontz

Contents

Abbreviations

ASD	anti-self-dual
BPST	Belavin, Polyakov, Schwartz, Tyupkin
EL	Euler–Lagrange
CR	Cauchy–Riemann
GR	general relativity
LIVF	left invariant vector field
ODE	ordinary differential equation
QCD	quantum chromodynamics
SD	self-dual
SI	Système International d'Unités

Chapter 1
Introduction

What I cannot create I do not understand.

— Richard Feynman in
"Feynman's Office: The Last Blackboards" [14]

This volume started out as notes in an attempt to help my students in a course on classical differential geometry. Not feeling constrained to use one particular text, I went off on my merry way introducing some of the basic structures of classical differential geometry (standard reference: [30]) that are used in physics. When the students requested specific references to texts, I would say that any one from a standard list of quite excellent texts would be fine. But my approach was not to be found in any *one* of them. Rather, the students had to search here and there in the literature and then try to piece it all together. So my notes were just that: a piecing together of things well known. However, in the spirit of the famous saying of Feynman (see Ref. [14]) noted above, some considerable part of the development of the subject is left for the reader to do in the exercises. Of course, in that same spirit, the reader should create explicitly all of the material presented here.

The main points of our approach to classical differential geometry follow:

- **Avoidance of explicit coordinates**
  This is a point of view advocated by Lang in [32]. We prefer to put the emphasis on what the coordinate functions really represent, namely, charts that are homeomorphisms onto open sets in a model space. This also eliminates an inundation of indices and so results in formulas that are much easier to understand. However, when we feel that a coordinate-free presentation is too cumbersome, we will use local coordinates.
- **The use of diagrams in lieu of formulas**
  This may seem to contradict the previous point but is meant to complement it. The point is that every formula can be expressed in words, but it is much preferable to use a formula and, among all possible formulas, the clearest of them all. (This is a sort of optimization problem whose study should be taken seriously!) But often the clearest formula can be rewritten as an even clearer diagram. And so we propose to do just that. An example of this is our definition of an Ehresmann connection, where we present the usual formulas but *also* their equivalent diagrams, which we find to be more transparent.

S.B. Sontz, *Principal Bundles*, Universitext, DOI 10.1007/978-3-319-14765-9_1

- **The definition of differential forms**
  We use a definition that explicitly displays 1-forms as dual maps (with values in the real line) to vector fields. Consequently, vector-valued differential forms are defined as maps with the same domains but with values in a given vector space.
- **The affine operation**
  We present the affine operation (also known as the translation function) as a basic structure defined on every Lie group G as well as on the fibers of every principal bundle with structure group G.
- **Early use of cocycles**
  We introduce at an early stage the concept of cocycle as the structure behind the construction of the tangent bundle and consequently in the definitions of vector bundle and principal bundle.
- **The importance of the fiber product**
  This is a well-known construction, but for whatever reason, it does not seem to be very popular in introductions to the subject matter. We find it to be extremely useful in our definition of differential forms k-forms for $k \geq 2$ and in the discussion of the affine operation.
- **The Maurer–Cartan form as an Ehresmann connection**
  The point here is that a seemingly trivial example of an Ehresmann connection gives us the definition of the Maurer–Cartan form of a Lie group. This leads to the natural problem of the classification of all Ehresmann connections on a Lie group considered as the total space of the principal bundle over a one-point space. This rather easy, yet rather important problem is rarely mentioned in introductory texts. But the answer illuminates the discussion of gauge fields and so deserves to be mentioned. In particular, proofs based on extended calculations are replaced by arguments based on the underlying structure of the theory, thereby revealing just why the calculations had to work out right.
- **The use of categories**
  We use category theory as an efficient language for describing many aspects of the theory. While this language is not absolutely necessary for understanding this book, this is a good time to learn it since it is essential language for the companion volume [44]. For reasons totally unclear to me, there still are some mathematicians who are allergic to this language and prefer to avoid it at any cost. As far as I am concerned, one might as well eliminate vector spaces as unnecessarily abstract constructs.
- **One definition of smooth**
  Paraphrasing Gertrude Stein, one of our maxims will be smooth is smooth is smooth. This means that we introduce one definition of a smooth manifold and one definition of a smooth map between smooth manifolds. While this seems to be a completely straightforward and uncontroversial application of the more general *categorical imperative*, it is too often violated in practice. For example, smooth distributions and smooth sections are sometimes given independent definitions as if they were not particular cases of the concept of a smooth map. We consider such an approach to be distracting, to say the least.

- **Not all concepts are defined**
  This might well be deemed heresy! Sometimes no definition is given at all since a lot of basic vocabulary is readily available in the 21st century on the Internet. Some terms that one would expect to be defined in 20th-century (or earlier) books are simply not defined. Just a few examples are linear, quotient space, and disjoint union.

  Of course, rigorous definitions are the essential first step of the basic *logical* process: definition, theorem, proof. However, the *mathematical* process begins with ideas, not definitions. Only then is the logical process followed. So some effort is made to explain clearly the ideas behind those definitions that are presented. But sometimes only an idea, even just a vague idea, is given and the definition, theorem, proof sequence is left as an exercise. This is part and parcel of the philosophy behind Feynman's maxim. This philosophy will be practiced in moderation. For example, it is used mostly in the discussion of results from category theory, which the reader need not dominate on a first reading. This will give the reader the opportunity to practice transforming ideas into rigorous definitions, which can then be used to prove theorems. The next step is to "learn" how to have ideas, but we leave that for another book.
- **Tangent vectors are underemphasized**
  If this had not been explicitly mentioned, the reader might never have noticed. Sure, the tangent vectors and tangent fields are here. So is the tangent bundle. But I deliberately only sketch the theory of vector fields and their integral curves, a standard topic! I am saying that these structures are not as central as they were once thought to be. (Heresy!) The important structures are their duals: cotangent vectors, 1-forms, and the cotangent bundle. But from our traditional, historically limited perspective, we do not understand how to grasp the dual structures directly as things in themselves. In this regard, we note that a generally accepted folklore of noncommutative spaces is that their "correct" infinitesimal structure is modeled in classical spaces by the de Rham theory, which is based on differential forms. In the companion volume [44], there will be little mention of tangents and their analogs in the noncommutative setting.

The reader may already be aware of the (whimsical!) definition of classical differential geometry as the study of properties invariant under change of notation. As with any pleasantry, there is a ring of truth to this. Our presentation suffers (as most others do) from a dependence on one system of notation. The novice should realize that further studies in this subject matter will necessarily include an inordinate amount of time spent in learning how to express things already learned in terms of another notation. Other introductions to these topics with their own notations and points of view exist in abundance, but I find some of them to be directed only at physicists wishing to learn some mathematics (just half of my point of view), some of them to be monographs rather than texts, and some to be unreadable for the typical beginner, whether physicist or mathematician. Of course, the encyclopedic two volumes [30] by S. Kobayashi and K. Nozimu are an excellent reference for differential geometry for those who have become more expert in the field.

So far I have mostly spoken about classical differential geometry. In [44], the sequel to this volume, I go on to present the more recent noncommutative theory. The word "classical" used to describe this volume is meant to contrast it with the theory in [44] of noncommutative geometry (also known as quantum geometry). This latter is again not original material, but the sources are different. Much of the motivation for noncommutative geometry is to be found in the classical theory even though the latter has an importance and an interest that extend further than its being a motivating factor for the newer noncommutative geometry. And this is why I start off with classical differential geometry and continue on to noncommutative geometry. However, a continuing challenge for research in noncommutative geometry is to find new concepts and applications that are more than mere imitations of the classical theory presented here.

We now present a brief outline of the contents of this book. Chapters 2 to 5 are essentially a translation of multivariable calculus into the modern language of differential geometry. In this sense, the last section of Chapter 5 is the denouement of that development. Of course, vector bundles, and more specifically the tangent bundle, are not much in evidence in the standard courses on multivariable calculus, but the point here is that they provide a concise language for discussing that material. A consequence is that there are no "real" theorems in these chapters. Rather, all the results are more or less consequences of the definitions. So the definitions are the nontrivial material in these chapters! An exception is the appendix to Chapter 3 on tensor products. This material is covered in many standard algebra courses, but in my experience, mathematics students often only know how to "add" vector spaces (direct sum) and not how to "multiply" them (tensor product). So I decided to include this material in an appendix for those who need to see it developed thoroughly. I imagine that many physics students will have to read this carefully too. Chapter 2 also has an appendix for those who need a more solid understanding of C^∞-functions, but proofs are not given.

Chapter 6 is an introduction to integral curves of vector fields and some of their applications. The existence and uniqueness of integral curves are shown to be equivalent to the same properties for an associated ordinary differential equation. This provides enough formalism to discuss the flows of electric and magnetic fields, which gets us into a bit of physics. The chapter concludes with the Lie derivative, very much a basic structure in differential geometry.

Chapter 7 is a frighteningly brief glimpse at the immense field of Lie group theory. It is hoped to be enough to get us through the theory of principal bundles, which is the main interest.

Next, Chapter 8 is mostly a cultural excursion into mathematically nontrivial territory, the Frobenius theorem. But we do motivate the definition of a distribution of subspaces in a tangent bundle. And this will be used later in the theory of connections. However, we do not prove the Frobenius theorem. Understanding what it says is plenty for now.

Chapters 2 to 8 provide the background material for launching into the topic of this book, principal bundles, which is dealt with in the remaining chapters. My intent has been to do this as quickly as is reasonably possible without skipping over the ever-important motivating intuitions. My more mathematically inclined readers may have seen some of that background already. Everyone is invited to skip sections or chapters that are already known, or mostly known. One can always refer back to skipped parts as need be. The rest of the book is meant to be a challenge for all of my potential readers. In some sense, the book really begins with Chapter 9. That is where the meat of the matter is begun to be presented. For my readers who do not have the prerequisites for understanding the prerequisite material before Chapter 9, please do not despair! There are more leisurely texts to get you up to speed. My personal favorite is Lee's text [33]. Actually, Lee proves a lot of the material that I present without proof. My view is that proofs can be important, but that understanding what is going on is even more important and can be sufficient, especially on a first pass through new material—but with the caveat that one has to really understand what is going on. Later on one can return to clean up the split infinitives that litter the way taken.

Chapters 9 to 11 provide a lot of material on principal bundles, their Ehresmann connections, and the curvature of those connections. These three chapters are the mathematical and geometrical core of the book.

Then Chapters 12 to 16 concern various applications of the general theory of principal bundles in physics. A lot of the exposition is aimed at providing physical motivation for people with backgrounds in mathematics. But there is also a lot of detailed description of how the physics is translated into the geometrical language of principal bundles for those with a physics background. So these chapters are not only aimed at all of my audience, but they also are meant to serve as an attempt to encourage more, and clearer, communication between these two communities. So it is the style of these chapters, even more than their well-known content, that gives this book whatever characteristic flavor it may have.

The text itself ends with Chapter 17, a one-page farewell to my kind readers, with some suggestions as to what they might want to do next with this knowledge. Of course, opening up new avenues of research not mentioned there would also be most welcome.

There is finally an appendix with discussions of almost all of the exercises. It is best to avoid looking at any of that, except after a long, hard consideration of an exercise. Remember Feynman's challenge!

Chapter 2
Basics of Manifolds

2.1 Charts

We start off with the idea of a chart. This is sometimes called a system of coordinates, but we feel, as does Lang (see [32]), that this both obscures the basic idea and impairs the recognition of an immediate generalization to Banach manifolds by introducing scads of unnecessary notation. The idea that Banach spaces provide an appropriate scenario for doing differential calculus goes back at least to the treatise [8] of Dieudonné. An extremely well-written text on smooth manifolds is Lee's book [33].

The setup is a Hausdorff topological space M.

Definition 2.1 *A* chart *in M with values in the vector space $\mathbb{R}^n$ (resp., in the Banach space B) is a pair (U, ϕ) such that U is an open subset of M in the topology of M and $\phi : U \to \phi(U) \subset \mathbb{R}^n$ is a homeomorphism onto an open set $\phi(U)$ in $\mathbb{R}^n$ [resp., $\phi : U \to \phi(U) \subset B$ is a homeomorphism onto an open set $\phi(U)$ in B]. Here $n \geq 0$ is an integer.*

Usually, we think about the case $n \geq 1$, though it is important to note that everything works (in a rather trivial way to be sure) when $n = 0$. We remind the reader that $\mathbb{R}^0 = \{0\}$, the trivial (but *not* empty) vector space.

Definition 2.2 *An* atlas *for M is a set of charts, say $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$, such that $\bigcup_\alpha U_\alpha = M$ and there is* one *integer n such that $\phi_\alpha(U_\alpha) \subset \mathbb{R}^n$ for all $\alpha \in A$ [resp., there is* one *Banach space B such that $\phi_\alpha(U_\alpha) \subset B$ for all $\alpha \in A$].*

From now on, we will consider the case where the *model space* is $\mathbb{R}^n$ rather than a general Banach space. However, the reader should bear in mind that a lot of this goes through *mutatis mutandis* with a Banach space B in place of $\mathbb{R}^n$. We remind the reader that there are (not finite-dimensional) Banach spaces that have no basis. So the generalization to manifolds with such a Banach space as the model space must not use coordinate notation. But somehow this fact is beside the point, which is simply to eliminate confusing, unneeded notation.

S.B. Sontz, *Principal Bundles*, Universitext, DOI 10.1007/978-3-319-14765-9_2

So the situation of interest for us is a topological space M such that every point has a neighborhood that looks like an open set in some (fixed) Euclidean space. One says: M is *locally Euclidean*. The existence of an atlas already greatly restricts the sort of topological spaces under consideration. However, one can impose more structure on the atlas and, consequently, further restrict the class of topological spaces being studied.

Definition 2.3 *We say that an atlas for M is a C^k*-atlas *(or that it is* of class C^k*) provided that for each pair of charts, say (U_α, ϕ_α) and (U_β, ϕ_β), in the atlas we have that the map*

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta) \tag{2.1}$$

is a function of class C^k. Furthermore, we say that $\phi_\beta \circ \phi_\alpha^{-1}$ is the transition function *(or the* change of coordinates *or the* change of charts*) from the chart (U_α, ϕ_α) to the chart (U_β, ϕ_β).*

Several comments are in order here.

- First, let us note that both $\phi_\alpha(U_\alpha \cap U_\beta)$ and $\phi_\beta(U_\alpha \cap U_\beta)$ are open subsets of $\mathbb{R}^n$. So when we say that $\phi_\beta \circ \phi_\alpha^{-1}$ is a function of class C^k here, we mean that in the sense of multivariable calculus. See the appendix to this chapter for more details on the class C^k.
- Second, in spite of the usual insistence on using different notation for a function and for its restriction to a subdomain, we are using ϕ_α both for the function in the chart (U_α, ϕ_α), that is,

$$\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha), \tag{2.2}$$

as well as for its restriction

$$\phi_\alpha : U_\alpha \cap U_\beta \to \phi_\alpha(U_\alpha \cap U_\beta), \tag{2.3}$$

which, by the way, makes all the sense in the world even when $U_\alpha \cap U_\beta$ is empty. Each of these versions of ϕ_α (namely, as given in (2.2) and in (2.3)) is a homeomorphism between its domain and its codomain. Analogous comments hold, of course, for our usage of the notation ϕ_β. This is how to interpret these maps in (2.1):

$$\phi_\alpha(U_\alpha \cap U_\beta) \xrightarrow{\phi_\alpha^{-1}} U_\alpha \cap U_\beta \xrightarrow{\phi_\beta} \phi_\beta(U_\alpha \cap U_\beta).$$

- Finally, we note that $k \geq 1$ is an integer or $k = \infty$. Most results hold for $k \geq 1$, though some only work for $k \geq 2$. However, we will only discuss the case $k = \infty$ here unless otherwise indicated. So we will drop the prefix C^k and simply say *atlas* when $k = \infty$.
  We use *smooth* as a synonym for class C^∞.

Definition 2.4 *We say a chart* (U, ϕ) *in* M *is* compatible *with a* C^k*-atlas* $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ *for* M *if the union,*

$$\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A} \cup (U, \phi),$$

is again a C^k*-atlas on* M.

So we throw one more chart into the atlas to get a new set of charts. Then we demand that this new set of charts be a C^k-atlas on M. In particular, $\phi(U)$ will be an open set in the same model space that is being used for the original atlas.

Exercise 2.1 *This definition is equivalent to demanding the condition that* $\phi \circ \phi_\alpha^{-1}$ *and* $\phi_\alpha \circ \phi^{-1}$ *be of class* C^k *for every* $\alpha \in A$.

We can introduce a partial order on atlases in the following way.

Definition 2.5 *Suppose that* $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ *and* $\mathcal{V} = \{(V_\gamma, \psi_\gamma)\}_{\gamma \in \Gamma}$ *are atlases on* M*. Then we write* $\mathcal{U} \ll \mathcal{V}$ *if each chart* (U_α, ϕ_α) *in* $\mathcal{U}$ *is also a chart in* $\mathcal{V}$*, that is, if there exists a* $\gamma \in \Gamma$ *such that* $(U_\alpha, \phi_\alpha) = (V_\gamma, \psi_\gamma)$*. Here equality means equality; that is,* $U_\alpha = V_\gamma$ *(the* same *open subset of* M*) and* $\phi_\alpha = \psi_\gamma$ *(the* same *function with the* same *domain and the* same *codomain).*

As a preliminary to the definition of differential manifold, we will use the following.

Definition 2.6 *A* differential structure *on a Hausdorff topological space* M *is a maximal smooth atlas, that is, a* C^∞*-atlas that is maximal among all such atlases with respect to the partial order* $\ll$.

Theorem 2.1 *Every smooth atlas* $\mathcal{U}$ *on* M *determines a unique differential structure on* M *containing it.*

Notice that we are *not* saying that a given Hausdorff topological space that has an atlas actually has a *smooth* atlas. (Indeed, there do exist Hausdorff topological spaces that have an atlas but do not have any smooth atlases. See [29] and [43]. We note in passing that a Hausdorff topological space that has an atlas is called a *topological manifold*.) We are saying that if a Hausdorff topological space does have a smooth atlas, then it has an associated differential structure.

Exercise 2.2 *Prove Theorem 2.1.*

2.2 The Objects - Differential Manifolds

Next, we define one of the central concepts of this theory.

Definition 2.7 *A* differential manifold *is a Hausdorff topological space* M *together with a given differential structure. This is also called a* smooth manifold*. Also, we define the* dimension of M *to be* n*, where* $\mathbb{R}^n$ *is the model space for* M*. The notation for this is* $\dim M = n$.