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Preface

The purpose of this book is to help students understand and use the calculus. Everything has been aimed toward making this easier, especially for students with limited background in mathematics or for readers who have forgotten their earlier training in mathematics. The topics covered include all the material of standard courses in elementary and intermediate calculus. The direct and concise exposition typical of the Schaum Outline series has been amplified by a large number of examples, followed by many carefully solved problems. In choosing these problems, we have attempted to anticipate the difficulties that normally beset the beginner. In addition, each chapter concludes with a collection of supplementary exercises with answers.

This sixth edition has enlarged the number of solved problems and supplementary exercises. Moreover, we have made a great effort to go over ticklish points of algebra or geometry that are likely to confuse the student. The author believes that most of the mistakes that students make in a calculus course are not due to a deficient comprehension of the principles of calculus, but rather to their weakness in high-school algebra or geometry. Students are urged to continue the study of each chapter until they are confident about their mastery of the material. A good test of that accomplishment would be their ability to answer the supplementary problems.

The author would like to thank many people who have written to me with corrections and suggestions, in particular Danielle Cinq-Mars, Lawrence Collins, L.D. De Jonge, Konrad Duch, Stephanie Happ, Lindsey Oh, and Stephen B. Soffer. He is also grateful to his editor, Charles Wall, for all his patient help and guidance.

ELLIOTT MENDELSON

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CHAPTER 1

Linear Coordinate Systems. Absolute Value. Inequalities

Linear Coordinate System

A linear coordinate system is a graphical representation of the real numbers as the points of a straight line. To each number corresponds one and only one point, and to each point corresponds one and only one number.

To set up a linear coordinate system on a given line: (1) select any point of the line as the *origin* and let that point correspond to the number 0; (2) choose a positive direction on the line and indicate that direction by an arrow; (3) choose a fixed distance as a unit of measure. If x is a positive number, find the point corresponding to x by moving a distance of x units from the origin in the positive direction. If x is negative, find the point corresponding to x by moving a distance of $-x$ units from the origin in the negative direction. (For example, if $x = -2$, then $-x = 2$ and the corresponding point lies 2 units from the origin in the negative direction.) See Fig. 1-1.

Fig. 1-1

The number assigned to a point by a coordinate system is called the *coordinate* of that point. We often will talk as if there is no distinction between a point and its coordinate. Thus, we might refer to "the point 3" rather than to "the point with coordinate 3."

The absolute value $|x|$ of a number x is defined as follows:

$$|x| = \begin{cases} x & \text{if } x \text{ is zero or a positive number} \\ -x & \text{if } x \text{ is a negative number} \end{cases}$$

For example, $|4| = 4$, $|-3| = -(-3) = 3$, and $|0| = 0$. Notice that, if x is a negative number, then $-x$ is positive. Thus, $|x| \geq 0$ for all x.

The following properties hold for any numbers x and y.

(1.1) $|-x| = |x|$
When $x = 0$, $|-x| = |-0| = |0| = |x|$.
When $x > 0$, $-x < 0$ and $|-x| = -(-x) = x = |x|$.
When $x < 0$, $-x > 0$, and $|-x| = -x = |x|$.

(1.2) $|x - y| = |y - x|$
This follows from (**1.1**), since $y - x = -(x - y)$.

(1.3) $|x| = c$ implies that $x = \pm c$.
For example, if $|x| = 2$, then $x = \pm 2$. For the proof, assume $|x| = c$.
If $x \geq 0$, $x = |x| = c$. If $x < 0$, $-x = |x| = c$; then $x = -(-x) = -c$.

(1.4) $|x|^2 = x^2$
If $x \geq 0$, $|x| = x$ and $|x|^2 = x^2$. If $x \leq 0$, $|x| = -x$ and $|x|^2 = (-x)^2 = x^2$.

(1.5) $|xy| = |x| \cdot |y|$
By (**1.4**), $|xy|^2 = (xy)^2 = x^2y^2 = |x|^2|y|^2 = (|x| \cdot |y|)^2$. Since absolute values are nonnegative, taking square roots yields $|xy| = |x| \cdot |y|$.

(1.6) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$ if $y \neq 0$

By (**1.5**), $|y|\left|\dfrac{x}{y}\right| = \left|y \cdot \dfrac{x}{y}\right| = |x|$. Divide by $|y|$.

(1.7) $|x| = |y|$ implies that $x = \pm y$

Assume $|x| = |y|$. If $y = 0$, $|x| = |0| = 0$ and (**1.3**) yields $x = 0$. If $y \neq 0$, then by (**1.6**),

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|} = 1$$

So, by (**1.3**), $x/y = \pm 1$. Hence, $x = \pm y$.

(1.8) Let $c \geq 0$. Then $|x| \leq c$ if and only if $-c \leq x \leq c$. See Fig. 1-2.

Assume $x \geq 0$. Then $|x| = x$. Also, since $c \geq 0$, $-c \leq 0 \leq x$. So, $|x| \leq c$ if and only if $-c \leq x \leq c$. Now assume $x < 0$. Then $|x| = -x$. Also, $x < 0 \leq c$. Moreover, $-x \leq c$ if and only if $-c \leq x$. (Multiplying or dividing an equality by a negative number reverses the inequality.) Hence, $|x| \leq c$ if and only if $-c \leq x \leq c$.

(1.9) Let $c \geq 0$. Then $|x| < c$ if and only if $-c < x < c$. See Fig. 1-2. The reasoning here is similar to that for (**1.8**).

Fig. 1-2

(1.10) $-|x| \leq x \leq |x|$

If $x \geq 0$, $x = |x|$. If $x < 0$, $|x| = -x$ and, therefore, $x = -|x|$.

(1.11) $|x + y| \leq |x| + |y|$ (triangle inequality)

By (**1.8**), $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. Adding, we obtain $-(|x| + |y|) \leq x + y \leq |x| + |y|$. Then $|x + y| \leq |x| + |y|$ by (**1.8**). [In (**1.8**), replace c by $|x| + |y|$ and x by $x + y$.]

Let a coordinate system be given on a line. Let P_1 and P_2 be points on the line having coordinates x_1 and x_2. See Fig. 1-3. Then:

(1.12) $|x_1 - x_2| = P_1P_2 =$ distance between P_1 and P_2.

This is clear when $0 < x_1 < x_2$ and when $x_1 < x_2 < 0$. When $x_1 < 0 < x_2$, and if we denote the origin by O, then $P_1P_2 = P_1O + OP_2 = (-x_1) + x_2 = x_2 - x_1 = |x_2 - x_1| = |x_1 - x_2|$.

As a special case of (**1.12**), when P_2 is the origin (and $x_2 = 0$):

(1.13) $|x_1| =$ distance between P_1 and the origin.

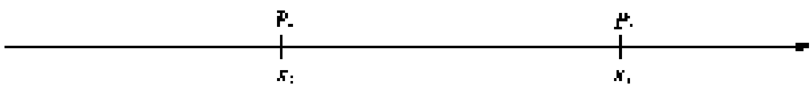


Fig. 1-3

Finite Intervals

Let $a < b$.

The *open interval* (a, b) is defined to be the set of all numbers between a and b, that is, the set of all x such that $a < x < b$. We shall use the term *open interval* and the notation (a, b) also for all the points between the points with coordinates a and b on a line. Notice that the open interval (a, b) does not contain the *endpoints* a and b. See Fig. 1-4.

The *closed interval* $[a, b]$ is defined to be the set of all numbers between a and b or equal to a or b, that is, the set of all x such that $a \leq x \leq b$. As in the case of open intervals, we extend the terminology and notation to points. Notice that the closed interval $[a, b]$ contains both endpoints a and b. See Fig. 1-4.

Fig. 1-4

By a *half-open interval* we mean an open interval (a, b) together with one of its endpoints. There are two such intervals: $[a, b)$ is the set of all x such that $a \le x < b$, and $(a, b]$ is the set of all x such that $a < x \le b$.

Infinite Intervals

Let (a, ∞) denote the set of all x such that $a < x$.
Let $[a, \infty)$ denote the set of all x such that $a \le x$.
Let $(-\infty, b)$ denote the set of all x such that $x < b$.
Let $(-\infty, b]$ denote the set of all x such that $x \le b$.

Inequalities

Any inequality, such as $2x - 3 > 0$ or $5 < 3x + 10 \le 16$, determines an interval. To solve an inequality means to determine the corresponding interval of numbers that satisfy the inequality.

EXAMPLE 1.1: Solve $2x - 3 > 0$.

$$2x - 3 > 0$$
$$2x > 3 \quad \text{(Adding 3)}$$
$$x > \tfrac{3}{2} \quad \text{(Dividing by 2)}$$

Thus, the corresponding interval is $(\tfrac{3}{2}, \infty)$.

EXAMPLE 1.2: Solve $5 < 3x + 10 \le 16$.

$$5 < 3x + 10 \le 16$$
$$-5 < 3x \le 6 \quad \text{(Subtracting 10)}$$
$$-\tfrac{5}{3} < x \le 2 \quad \text{(Dividing by 3)}$$

Thus, the corresponding interval is $(-\tfrac{5}{3}, 2]$.

EXAMPLE 1.3: Solve $-2x + 3 < 7$.

$$-2x + 3 < 7$$
$$-2x < 4 \quad \text{(Subtracting 3)}$$
$$x > -2 \quad \text{(Dividing by } -2\text{)}$$

(Recall that dividing by a negative number reverses an inequality.) Thus, the corresponding interval is $(-2, \infty)$.

SOLVED PROBLEMS

1. Describe and diagram the following intervals, and write their interval notation, (a) $-3 < x < 5$; (*b*) $2 \le x \le 6$; (c) $-4 < x \le 0$; (d) $x > 5$; (e) $x \le 2$; (f) $3x - 4 \le 8$; (g) $1 < 5 - 3x < 11$.

(a) All numbers greater than -3 and less than 5; the interval notation is $(-3, 5)$:

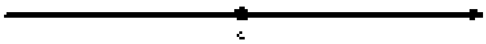
(b) All numbers equal to or greater than 2 and less than or equal to 6; [2, 6]:

(c) All numbers greater than −4 and less than or equal to 0; (−4, 0]:

(d) All numbers greater than 5; (5, ∞):

(e) All numbers less than or equal to 2; (−∞, 2]:

(f) $3x - 4 \le 8$ is equivalent to $3x \le 12$ and, therefore, to $x \le 4$. Thus, we get (−∞, 4]:



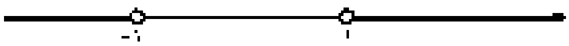
(g)
$$1 < 5 - 3x < 11$$
$$-4 < -3x < 6 \quad \text{(Subtracting 5)}$$
$$-2 < x < \tfrac{4}{3} \quad \text{(Dividing by } -3\text{; note the reversal of inequalities)}$$

Thus, we obtain $(-2, \frac{4}{3})$:

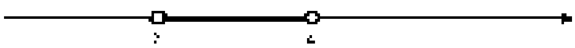
2. Describe and diagram the intervals determined by the following inequalities, (a) $|x| < 2$; (b) $|x| > 3$; (c) $|x - 3| < 1$; (d) $|x - 2| < \delta$ where $\delta > 0$; (e) $|x + 2| \le 3$; (f) $0 < |x - 4| < \delta$ where $\delta > 0$.

(a) By property (**1.9**), this is equivalent to $-2 < x < 2$, defining the open interval (−2, 2).

(b) By property (**1.8**), $|x| \le 3$ is equivalent to $-3 \le x \le 3$. Taking negations, $|x| > 3$ is equivalent to $x < -3$ or $x > 3$, which defines the union of the intervals (−∞, −3) and (3, ∞).



(c) By property (**1.12**), this says that the distance between x and 3 is less than 1, which is equivalent to $2 < x < 4$. This defines the open interval (2, 4).



We can also note that $|x - 3| < 1$ is equivalent to $-1 < x - 3 < 1$. Adding 3, we obtain $2 < x < 4$.

(d) This is equivalent to saying that the distance between x and 2 is less than δ, or that $2 - \delta < x < 2 + \delta$, which defines the open interval $(2 - \delta, 2 + \delta)$. This interval is called the *δ-neighborhood* of 2:

(e) $|x + 2| < 3$ is equivalent to $-3 < x + 2 < 3$. Subtracting 2, we obtain $-5 < x < 1$, which defines the open interval $(-5, 1)$:

(f) The inequality $|x - 4| < \delta$ determines the interval $4 - \delta < x < 4 + \delta$. The additional condition $0 < |x - 4|$ tells us that $x \neq 4$. Thus, we get the union of the two intervals $(4 - \delta, 4)$ and $(4, 4 + \delta)$. The result is called the *deleted δ-neighborhood* of 4:

3. Describe and diagram the intervals determined by the following inequalities, (a) $|5 - x| \leq 3$; (b) $|2x - 3| < 5$; (c) $|1 - 4x| < \frac{1}{2}$.

(a) Since $|5 - x| = |x - 5|$, we have $|x - 5| \leq 3$, which is equivalent to $-3 \leq x - 5 \leq 3$. Adding 5, we get $2 \leq x \leq 8$, which defines the closed interval $[2, 8]$:

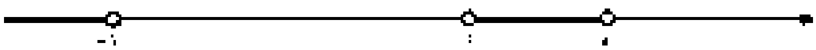
(b) $|2x - 3| < 5$ is equivalent to $-5 < 2x - 3 < 5$. Adding 3, we have $-2 < 2x < 8$; then dividing by 2 yields $-1 < x < 4$, which defines the open interval $(-1, 4)$:

(c) Since $|1 - 4x| = |4x - 1|$, we have $|4x - 1| < \frac{1}{2}$, which is equivalent to $-\frac{1}{2} < 4x - 1 < \frac{1}{2}$. Adding 1, we get $\frac{1}{2} < 4x < \frac{3}{2}$. Dividing by 4, we obtain $\frac{1}{8} < x < \frac{3}{8}$, which defines the open interval $(\frac{1}{8}, \frac{3}{8})$:

4. Solve the inequalities: (a) $18x - 3x^2 > 0$; (b) $(x + 3)(x - 2)(x - 4) < 0$; (c) $(x + 1)^2(x - 3) > 0$, and diagram the solutions.

(a) Set $18x - 3x^2 = 3x(6 - x) = 0$, obtaining $x = 0$ and $x = 6$. We need to determine the sign of $18x - 3x^2$ on each of the intervals $x < 0$, $0 < x < 6$, and $x > 6$, to determine where $18x - 3x^2 > 0$. Note that it is negative when $x < 0$ (since x is negative and $6 - x$ is positive). It becomes positive when we pass from left to right through 0 (since x changes sign but $6 - x$ remains positive), and it becomes negative when we pass through 6 (since x remains positive but $6 - x$ changes to negative). Hence, it is positive when and only when $0 < x < 6$.

(b) The crucial points are $x = -3$, $x = 2$, and $x = 4$. Note that $(x + 3)(x - 2)(x - 4)$ is negative for $x < -3$ (since each of the factors is negative) and that it changes sign when we pass through each of the crucial points. Hence, it is negative for $x < -3$ and for $2 < x < 4$:



(c) Note that $(x + 1)$ is always positive (except at $x = -1$, where it is 0). Hence $(x + 1)^2 (x - 3) > 0$ when and only when $x - 3 > 0$, that is, for $x > 3$:

5. Solve $|3x - 7| = 8$.

By (**1.3**), $|3x - 7| = 8$ if and only if $3x - 7 = \pm 8$. Thus, we need to solve $3x - 7 = 8$ and $3x - 7 = -8$. Hence, we get $x = 5$ or $x = -\frac{1}{3}$.

6. Solve $\dfrac{2x+1}{x+3} > 3$.

Case 1: $x + 3 > 0$. Multiply by $x + 3$ to obtain $2x + 1 > 3x + 9$, which reduces to $-8 > x$. However, since $x + 3 > 0$, it must be that $x > -3$. Thus, this case yields no solutions.

Case 2: $x + 3 < 0$. Multiply by $x + 3$ to obtain $2x + 1 < 3x + 9$. (Note that the inequality is reversed, since we multiplied by a negative number.) This yields $-8 < x$. Since $x + 3 < 0$, we have $x < -3$. Thus, the only solutions are $-8 < x < -3$.

7. Solve $\left|\dfrac{2}{x} - 3\right| < 5$.

The given inequality is equivalent to $-5 < \dfrac{2}{x} - 3 < 5$. Add 3 to obtain $-2 < 2/x < 8$, and divide by 2 to get $-1 < 1/x < 4$.

Case 1: $x > 0$. Multiply by x to get $-x < 1 < 4x$. Then $x > \frac{1}{4}$ and $x > -1$; these two inequalities are equivalent to the single inequality $x > \frac{1}{4}$.

Case 2: $x < 0$. Multiply by x to obtain $-x > 1 > 4x$. (Note that the inequalities have been reversed, since we multiplied by the negative number x.) Then $x < \frac{1}{4}$ and $x < -1$. These two inequalities are equivalent to $x < -1$.

Thus, the solutions are $x > \frac{1}{4}$ or $x < -1$, the union of the two infinite intervals $(\frac{1}{4}, \infty)$ and $(-\infty, -1)$.

8. Solve $|2x - 5| \geq 3$.

Let us first solve the negation $|2x - 5| < 3$. The latter is equivalent to $-3 < 2x - 5 < 3$. Add 5 to obtain $2 < 2x < 8$, and divide by 2 to obtain $1 < x < 4$. Since this is the solution of the negation, the original inequality has the solution $x \leq 1$ or $x \geq 4$.

9. Solve: $x^2 < 3x + 10$.

$$\begin{aligned} &x^2 < 3x + 10 \\ &x^2 - 3x - 10 < 0 \qquad \text{(Subtract } 3x + 10\text{)} \\ &(x - 5)(x + 2) < 0 \end{aligned}$$

The crucial numbers are -2 and 5. $(x - 5)(x + 2) > 0$ when $x < -2$ (since both $x - 5$ and $x + 2$ are negative); it becomes negative as we pass through -2 (since $x + 2$ changes sign); and then it becomes positive as we pass through 5 (since $x - 5$ changes sign). Thus, the solutions are $-2 < x < 5$.

SUPPLEMENTARY PROBLEMS

10. Describe and diagram the set determined by each of the following conditions:

(a) $-5 < x < 0$
(b) $x \leq 0$
(c) $-2 \leq x < 3$
(d) $x \geq 1$
(e) $|x| < 3$
(f) $|x| \geq 5$
(g) $|x - 2| < \frac{1}{2}$
(h) $|x - 3| > 1$
(i) $0 < |x - 2| < 1$
(j) $0 < |x + 3| < \frac{1}{4}$
(k) $|x - 2| \geq 1$.

Ans. (e) $-3 < x < 3$; (f) $x \geq 5$ or $x \leq -5$; (g) $\frac{3}{2} < x < \frac{5}{2}$; (h) $x > -2$ or $x < -4$; (i) $x \neq 2$ and $1 < x < 3$; (j) $-\frac{13}{4} < x < -\frac{11}{4}$; (k) $x \geq 3$ or $x \leq 1$

11. Describe and diagram the set determined by each of the following conditions:

(a) $|3x - 7| < 2$
(b) $|4x - 1| \geq 1$
(c) $\left|\dfrac{x}{3} - 2\right| \leq 4$

(d) $\left|\frac{3}{x} - 2\right| \le 4$

(e) $\left|2 + \frac{1}{x}\right| > 1$

(f) $\left|\frac{4}{x}\right| < 3$

Ans. (a) $\frac{5}{3} < x < 3$; (b) $x \ge \frac{1}{2}$ or $x \le 0$; (c) $-6 \le x \le 18$; (d) $x \le -\frac{3}{2}$ or $x \ge \frac{1}{2}$; (e) $x > 0$ or $x < -1$ or $-\frac{1}{3} < x < 0$; (f) $x > \frac{4}{3}$ or $x < -\frac{4}{3}$

12. Describe and diagram the set determined by each of the following conditions:

(a) $x(x-5) < 0$
(b) $(x-2)(x-6) > 0$
(c) $(x+1)(x-2) < 0$
(d) $x(x-2)(x+3) > 0$
(e) $(x+2)(x+3)(x+4) < 0$
(f) $(x-1)(x+1)(x-2)(x+3) > 0$
(g) $(x-1)^2(x+4) > 0$
(h) $(x-3)(x+5)(x-4)^2 < 0$
(i) $(x-2)^3 > 0$
(j) $(x+1)^3 < 0$
(k) $(x-2)^3(x+1) < 0$
(l) $(x-1)^3(x+1)^4 < 0$
(m) $(3x-1)(2x+3) > 0$
(n) $(x-4)(2x-3) < 0$

Ans. (a) $0 < x < 5$; (b) $x > 6$ or $x < 2$; (c) $-1 < x < 2$; (d) $x > 2$ or $-3 < x < 0$; (e) $-3 < x < -2$ or $x < -4$; (f) $x > 2$ or $-1 < x < 1$ or $x < -3$; (g) $x > -4$ and $x \ne 1$; (h) $-5 < x < 3$; (i) $x > 2$; (j) $x < -1$; (k) $-1 < x < 2$; (l) $x < 1$ and $x \ne -1$; (m) $x > \frac{1}{3}$ or $x < -\frac{3}{2}$; (n) $\frac{3}{2} < x < 4$

13. Describe and diagram the set determined by each of the following conditions:

(a) $x^2 < 4$
(b) $x^2 \ge 9$
(c) $(x-2)^2 \le 16$
(d) $(2x+1)^2 > 1$
(e) $x^2 + 3x - 4 > 0$
(f) $x^2 + 6x + 8 \le 0$
(g) $x^2 < 5x + 14$
(h) $2x^2 > x + 6$
(i) $6x^2 + 13x < 5$
(j) $x^3 + 3x^2 > 10x$

Ans. (a) $-2 < x < 2$; (b) $x \ge 3$ or $x \le -3$; (c) $-2 \le x \le 6$; (d) $x > 0$ or $x < -1$; (e) $x > 1$ or $x < -4$; (f) $-4 \le x \le -2$; (g) $-2 < x < 7$; (h) $x > 2$ or $x < -\frac{3}{2}$; (i) $-\frac{5}{2} < x < \frac{1}{3}$; (j) $-5 < x < 0$ or $x > 2$

14. Solve: (a) $-4 < 2 - x < 7$ (b) $\frac{2x-1}{x} < 3$ (c) $\frac{x}{x+2} < 1$

(d) $\frac{3x-1}{2x+3} > 3$ (e) $\left|\frac{2x-1}{x}\right| > 2$ (f) $\left|\frac{x}{x+2}\right| \le 2$

Ans. (a) $-5 < x < 6$; (b) $x > 0$ or $x < -1$; (c) $x > -2$; (d) $-\frac{10}{3} < x < \frac{3}{2}$; (e) $x < 0$ or $0 < x < \frac{1}{4}$; (f) $x \le -4$ or $x \ge -1$

15. Solve:

(a) $|4x - 5| = 3$
(b) $|x + 6| = 2$
(c) $|3x - 4| = |2x + 1|$
(d) $|x + 1| = |x + 2|$
(e) $|x + 1| = 3x - 1$
(f) $|x + 1| < |3x - 1|$
(g) $|3x - 4| \geq |2x + 1|$

Ans. (a) $x = 2$ or $x = \frac{1}{2}$; (b) $x = -4$ or $x = -8$; (c) $x = 5$ or $x = \frac{5}{3}$; (d) $x = -\frac{3}{2}$; (e) $x = 1$; (f) $x > 1$ or $x < 0$; (g) $x \geq 5$ or $x \leq \frac{3}{5}$

16. Prove:

(a) $|x^2| = |x|^2$;
(b) $|x^n| = |x|^n$ for every integer n;
(c) $|x| = \sqrt{x^2}$;
(d) $|x - y| \leq |x| + |y|$;
(e) $|x - y| \geq ||x| - |y||$
[Hint: In (e), prove that $|x - y| \geq |x| - |y|$ and $|x - y| \geq |y| - |x|$.]

CHAPTER 2

Rectangular Coordinate Systems

Coordinate Axes

In any plane $\mathcal{P}$, choose a pair of perpendicular lines. Let one of the lines be horizontal. Then the other line must be vertical. The horizontal line is called the *x axis*, and the vertical line the *y axis*. (See Fig. 2-1.)

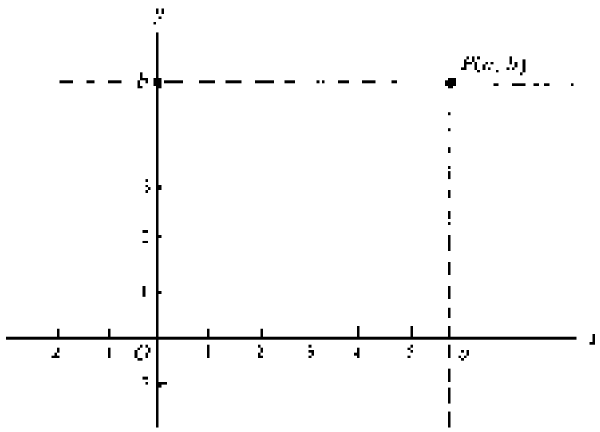


Fig. 2-1

Now choose linear coordinate systems on the x axis and the y axis satisfying the following conditions: The origin for each coordinate system is the point O at which the axes intersect. The x axis is directed from left to right, and the y axis from bottom to top. The part of the x axis with positive coordinates is called the *positive x axis*, and the part of the y axis with positive coordinates is called the *positive y axis*.

We shall establish a correspondence between the points of the plane $\mathcal{P}$ and pairs of real numbers.

Coordinates

Consider any point P of the plane (Fig. 2-1). The vertical line through P intersects the x axis at a unique point; let a be the coordinate of this point on the x axis. The number a is called the *x coordinate* of P (or the *abscissa* of P). The horizontal line through P intersects the y axis at a unique point; let b be the coordinate of this point on the y axis. The number b is called the *y coordinate* of P (or the *ordinate* of P). In this way, every point P has a unique pair (a, b) of real numbers associated with it. Conversely, every pair (a, b) of real numbers is associated with a unique point in the plane.

The coordinates of several points are shown in Fig. 2-2. For the sake of simplicity, we have limited them to integers.

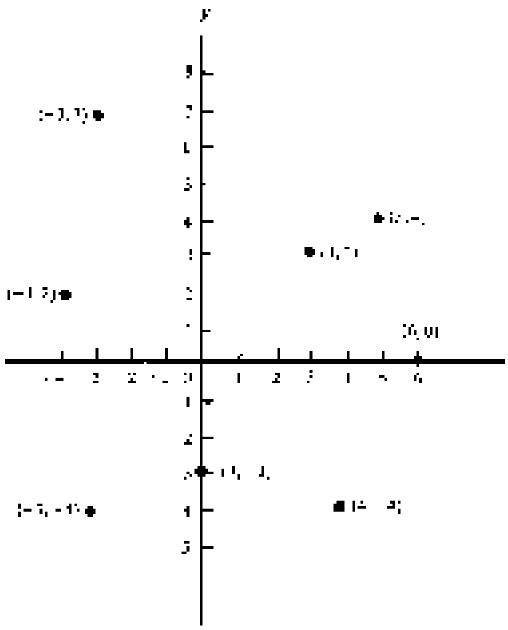


Fig. 2-2

EXAMPLE 2.1: In the coordinate system of Fig. 2-3, to find the point having coordinates (2, 3), start at the origin, move two units to the *right*, and then three units *upward*.

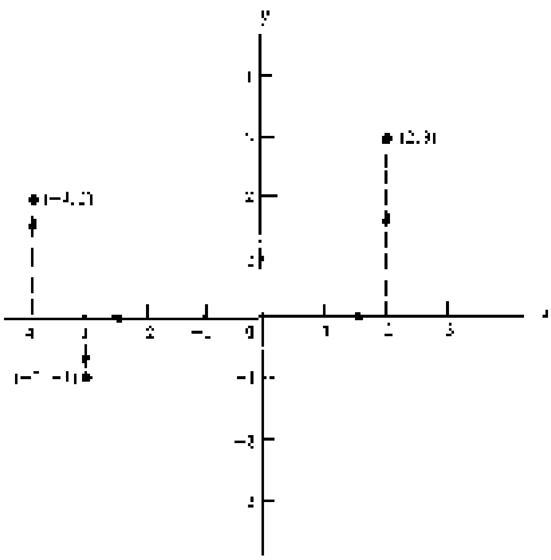


Fig. 2-3

To find the point with coordinates (–4, 2), start at the origin, move four units to the *left*, and then two units *upward*.

To find the point with coordinates (–3, –1), start at the origin, move three units to the *left*, and then one unit *downward*.

The order of these moves is not important. Hence, for example, the point (2, 3) can also be reached by starting at the origin, moving three units *upward*, and then two units to the *right*.

Quadrants

Assume that a coordinate system has been established in the plane $\mathcal{P}$. Then the whole plane $\mathcal{P}$, with the exception of the coordinate axes, can be divided into four equal parts, called *quadrants*. All points with both coordinates positive form the first quadrant, called quadrant I, in the upper-right-hand corner (see Fig. 2-4).